FUNCTORS FOR GENUINE REPRESENTATIONS OF THE METAPLECTIC GROUP AND GRADED AFFINE HECKE ALGEBRAS

by

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ABSTRACT

In a recent pre-print, Ciubotaru and Trapa defined a family of exact functors carrying spherical Harish-Chandra modules for real classical linear algebraic groups to representations of a certain algebra called the graded affine Hecke algebra. Representations of this algebra can then be translated, thanks to results of Lusztig, Barbasch, and Moy, into representations of a p-adic group of the same type as the original real group. The result, in effect, is a Lefschetz functor for real classical linear algebraic groups; it also embeds the spherical unitary dual for the real group into the spherical unitary dual for the p-adic group. This thesis develops an analagous functor for genuine representations of the real and p-adic metaplectic groups.

To my father, who would have been very proud.

CONTENTS

NOTATION AND SYMBOLS

$\mathrm{Sp}(2n, F)$	The symplectic group over F.
$\widetilde{\mathrm{Sp}}(2n, F)$	The metaplectic group over F.
$\mathrm{U}(n)$	The unitary group.
$G_{\mathbb{R}}$	A split real linear algebraic group
$\widetilde{G}_{\mathbb{R}}$	The double cover of $G_{\mathbb{R}}$ (usually $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$)
$\mathcal{H}(G)$	An Iwahori-Hecke algebra for the group G.
$\mathbb{H}_R(\mathbf{c})$	The graded affine Hecke algebra for the root system R with parameter $\mathbf{c}$.
$I_\omega(\nu)$	A standard module for $\widetilde{\mathrm{Sp}}(2n, F)$ or $\mathrm{GL}(n, \mathbb{R})$
$J_\omega(\nu)$	The unique irreducible subquotient of $I_\omega(\nu)$ containing ω
$\mathbb{I}_1(\nu)$	A standard module for $\mathbb{H}_R(\mathbf{c})$
$\mathbb{J}_1(\nu)$	The unique irreducible spherical subquotient of $\mathbb{I}_1(\nu)$
$\mathcal{F}_{\mu_0, V, k}$	Ciubotaru and Trapa's Lefschetz functor for classical linear groups

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CHAPTER 1

INTRODUCTION

In a recent pre-print [7], Ciubotaru and Trapa defined a family of exact functors $\mathcal{F}_{\mu_0,V,k}$ carrying spherical Harish-Chandra modules for real classical linear algebraic groups to representations of a certain algebra called the graded affine Hecke algebra. Representations of this algebra can then be translated, thanks to results of Lusztig, Barbasch, and Moy, into representations of a p-adic group of the same type as the original real group [4] [13]. The result, in effect, is a Lefschetz functor for real classical linear algebraic groups.

The Lefschetz functors defined by Ciubotaru and Trapa have several very useful properties. In particular the functors carry spherical principal series representations of the real group to spherical principal series representations of the graded affine Hecke algebra, carry irreducible spherical representations to irreducible spherical representations, and carry irreducible unitary representations to irreducible unitary representations.

My thesis extends this work in two ways:

1. By developing an analogous functor for pseudospherical series representations of the metaplectic group.

2. By developing an analogous functor for nonspherical representations of $\mathrm{GL}(n, \mathbb{R})$.

It will be instructive to begin with a review of the results of Ciubotaru and Trapa.

1.1 Lefschetz Functors for Linear Groups

Suppose $G_{\mathbb{R}}$ is a split real linear algebraic group, regarded as a Chevalley group with real coefficients. Fix $G_{\mathbb{Q}_p}$ as the corresponding Chevalley group over $\mathbb{Q}_p$ (see for instance [16]). Ciubotaru and Trapa were interested in developing a functor $\mathcal{F}_{\mu_0,V,k}$ which consumes a spherical Harish-Chandra module for $G_{\mathbb{R}}$ and produces a representation of $G_{\mathbb{Q}_p}$.

The representation theory of p-adic groups is quite complicated, but an important fact about p-adic groups is that much of their representation theory is controlled by an algebra known as the Iwahori-Hecke algebra $\mathcal{H}(G_{\mathbb{Q}_p})$ (see section 2.2.1 for details). As a vector space, the Iwahori-Hecke algebra is simply the space of compactly supported smooth functions

$C_c(I\backslash G_{\mathbb{Q}_p}/I)$, bi-invariant under the action of a certain compact-open subgroup I, called an Iwahori subgroup. The product on this space is given by convolution. The Borel-Casselman correspondence (see [5]) states that

Theorem 1 *There is a categorical equivalence between the category of $\mathcal{H}(G_{\mathbb{Q}_p})$ modules and the category of admissable $G_{\mathbb{Q}_p}$ representations generated by their I-fixed vectors.*

Unfortunately the algebra $\mathcal{H}(G_{\mathbb{Q}_p})$ is still quite complicated. Fortunately, due to work of Lusztig [13], the representation theory of $\mathcal{H}(G_{\mathbb{Q}_p})$ can often be reduced to the representation theory of a far simpler algebra known as the graded affine Hecke algebra.

Definition 2 *Fix a complex vector space V, a root system $R \subset V^*$, its Weyl group W, and a W-invariant map $\mathbf{c} : R \longrightarrow \mathbb{Q}$. Write the action of $w \in W$ on $f \in V$ as $w \cdot f$. Then $\mathbb{H}_R(\mathbf{c})$ is the unique complex associative algebra such that:*

1. $\mathbb{H}_R(\mathbf{c}) \cong \mathbb{C}[W] \otimes S(V)$ *(as a vector space)*

2. $\mathbb{C}[W]$ *injects into the first factor and* $S(V)$ *injects into the second factor as algebras*

3. $(w \otimes 1)(1 \otimes f) = w \otimes f$ *(written wf)*

4. *If $\alpha \in R$, $f \in V$, and $s_\alpha \in W$ is the simple reflection through the root α, then*

$$s_\alpha f - (s_\alpha \cdot f)s_\alpha = \mathbf{c}(\alpha)\alpha(f)$$

Both $\mathcal{H}(G_{\mathbb{Q}_p})$ and $\mathbb{H}_R(\mathbf{c})$ inherit $*$-operations from $G_{\mathbb{Q}_p}$ which make it possible to define the notion of a unitary representation. Barbasch and Moy proved that if $G_{\mathbb{Q}_p}$ is a split adjoint group and $\mathbf{c}$ is constant, this series of correspondences, carrying representations of $G_{\mathbb{Q}_p}$ to $\mathbb{H}_R(\mathbf{c})$ modules and vice versa, preserve unitarity ([3] and [4]).

Ciubotaru's and Trapa's Lefschetz functors for linear groups actually carry Harish-Chandra modules for $G_{\mathbb{R}}$ to $\mathbb{H}_R(\mathbf{c})$-modules. Both of these categories of modules include particularly important objects called principal series representations.

Definition 3 *Suppose G is a split real group with maximal compact subgroup K, and B is a Borel subgroup with Langlands decomposition $B = MAN$. Denote the Lie algebra of A by $\mathfrak{a}$. Given a character ω of M and $\nu \in \mathfrak{a}^*$, the minimal principal series representations of G are defined to be*

$$I_\omega(\nu) = \mathrm{Ind}_B^G(\omega \otimes e^\nu \otimes 1)$$

where normalized induction is being used following [11]. Also denote by $J_\omega(\nu)$ *the unique irreducible subquotient of* $I_\omega(\nu)$ *containing the* K*-type* ω*. If* ω *is the trivial representation then these representations are said to be* spherical.

The objects in the category $\mathcal{HC}_1(\lambda, G)$ are subquotients of spherical minimal principal series representations of G with infinitesimal character λ.

The principal series representations of $\mathbb{H}_R(\mathbf{c})$ look similar:

Definition 4 *Given* $\nu \in V^*$*, denote by* $\mathbb{C}_\nu$ *the unique one-dimensional representation of* $S(V)$ *such that* $w \cdot 1 = \nu(w)$ *for all* $w \in V$*. Then the spherical principal series for* $\mathbb{H}_R(\mathbf{c})$ *is defined to be*

$$\mathbb{I}_1(\nu) = \mathbb{H}_R(\mathbf{c}) \otimes_{S(V)} \mathbb{C}_\nu$$

Denote the unique irreducible spherical subquotient of $\mathbb{I}_1(\nu)$ *by* $\mathbb{J}_1(\nu)$*.*

Ciubotaru and Trapa proved the following [7]:

Theorem 5 *Suppose* G *is the group* $\mathrm{GL}(n,\mathbb{R})$*,* $\mathrm{U}(p,q)$*,* $\mathrm{Sp}(2n,\mathbb{R})$*, or* $\mathrm{O}(p,q)$*. Denote by* V *the defining representation of* G*, by* K *its maximal compact subgroup, and by* k *the real rank of* G*. There is a choice of one-dimensional representation* μ_0 *of* K*, root system* R*, and parameter* $\mathbf{c}$ *such that for any object* X *in the category* $\mathcal{HC}_1(\lambda, G)$ *there is a natural action of* $\mathbb{H}_R(\mathbf{c})$ *on the space* $\mathrm{Hom}_{K_\mathbb{R}}\Big(\mu_0, X \otimes \big(V\big)^{\otimes k}\Big)$*. This defines an exact functor*

$$\mathcal{F}_{\mu_0,V,k} : \mathcal{HC}_1(\lambda, G) \longrightarrow \mathbb{H}_R(\mathbf{c})\text{-}mod$$

such that:

1. $\mathcal{F}_{\mu_0,V,k}(I_1(\lambda)) = \mathbb{I}_1(\lambda)$ *and* $\mathcal{F}_{\mu_0,V,k}(J_1(\lambda)) = \mathbb{J}_1(\lambda)$
2. *If* X *is an irreducible Hermitian (respectively unitary) object in* $\mathcal{HC}_1(\lambda, G)$ *then* $\mathcal{F}_{\mu_0,V,k}(X)$ *is an irreducible Hermitian (respectively unitary) object in* $\mathbb{H}_R(\mathbf{c})$*-mod*

This theorem embeds the spherical unitary dual for G into the spherical unitary dual of $\mathbb{H}_R(\mathbf{c})$.

CHAPTER 2

METAPLECTIC CASE

2.1 Notation

Before proceeding with the generalization to the Metaplectic group, it will be useful to recall the definition. Fix a matrix $J \in \mathrm{GL}(2n, \mathbb{R})$:

$$J = \begin{pmatrix} 0 & J' \\ -J' & 0 \end{pmatrix}$$

where J' is the matrix with 1 on the opposite-diagonal and 0 elsewhere. The lie group $\mathrm{Sp}(2n, \mathbb{R})$ is defined to be the subgroup of $\mathrm{GL}(2n, \mathbb{R})$ such that

$$g^{\mathrm{T}} J g = J \quad \forall g \in \mathrm{Sp}(2n, \mathbb{R})$$

The lie algebra $\mathfrak{sp}(2n, \mathbb{R})$ is defined by the derivative of this condition, i.e., that

$$X^{\mathrm{T}} J + JX = 0 \quad \forall X \in \mathfrak{sp}(2n, \mathbb{R}) \tag{2.1}$$

This condition will yield a sensible basis for $\mathfrak{sp}(2n, \mathbb{R})$. I will adopt the convention that $E_{i,j}$ is the matrix with 1 in the $(i, j)^{th}$ position and 0 elsewhere.

The metaplectic group $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ is the unique (up to isomorphism) nontrivial connected central extension of $\mathrm{Sp}(2n, \mathbb{R})$ by $\mathbb{Z}/2\mathbb{Z}$. Denote the projection from $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ to $\mathrm{Sp}(2n, \mathbb{R})$ by pr:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \widetilde{\mathrm{Sp}}(2n, \mathbb{R}) \xrightarrow{\mathrm{pr}} \mathrm{Sp}(2n, \mathbb{R}) \longrightarrow 0$$

For the remainder of this chapter, G will denote $\widetilde{\mathrm{Sp}}(2n, \mathbb{C})$, and $G_{\mathbb{R}}$ will denote $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$, regarded as a subgroup of G. Likewise, $\mathfrak{g}_{\mathbb{R}}$ will denote the Lie algebra of $G_{\mathbb{R}}$, and $\mathfrak{g}$ will denote its complexification, the Lie algebra of G.

The group $\mathrm{Sp}(2n, \mathbb{R})$ has maximal compact subgroup isomorphic to $\mathrm{U}(n)$; it is the maximal compact subgroup fixed by the involution $g \mapsto (\xi')^{-1} g \xi'$, where

$$\xi' = iJ$$

Note that on $\mathrm{Sp}(2n, \mathbb{R})$, this corresponds to the involution $X \mapsto -X^{\mathrm{T}}$.

The maximal compact subgroup of $\widetilde{K}_{\mathbb{R}}$ of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ is $\mathrm{pr}^{-1}(\mathrm{U}(n))$. This can be written explicitly by identifying it with a subgroup of $\mathrm{U}(n)\times\mathrm{U}(1)$ (see [14]):

$$K_{\mathbb{R}} = \{[g,z] \in \mathrm{U}(n)\times\mathrm{U}(1) | \det(g) = z^2\}$$

This is the maximal compact subgroup fixed by the involution $\theta : g \mapsto \xi g \xi^{-1}$, where $\xi \in \mathrm{pr}^{-1}(\xi')$; because $\mathrm{U}(1)$ is abelian, the precise choice of ξ doesn't matter. Like the involution above, this corresponds loosely to negative transposition; on $K_{\mathbb{R}}$ it corresponds specifically to $[k,z] \mapsto [-k^{\mathrm{T}}, z]$.

The involution θ is order 2, and divides $\mathfrak{g}$ into ± 1 eigenspaces. The $+1$ eigenspace is the complexified Lie algebra of K, denoted $\mathfrak{k}$, and the -1 eigenspace will be denoted $\mathfrak{p}$. We then have the Cartan decomposition:

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

The root system is specified by choosing a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$; this is a cartan subalgebra. For brevity I will use the notation that $\sigma(i) = 2n+1-i$. In this paper the cartan subalgebra in question will be of the form

$$\mathfrak{a} = \mathrm{span}\left\{E_{i,i} - E_{\sigma(i),\sigma(i)} \,|\, 1 \le i \le n\right\}$$

The choice of positive roots is

$$\mathfrak{n} = \mathrm{span}\left\{\sqrt{2}E_{i,\sigma(i)}, E_{i,j} - E_{\sigma(j),\sigma(i)}, E_{i,\sigma(j)} + E_{j,\sigma(i)} \,|\, 1 \le i < j \le n\right\}$$

Note that $\{E_{i,i+1} - E_{\sigma(i),\sigma(i+1)}\}$ are the short simple roots, and $\sqrt{2}E_{1,\sigma(1)}$ is the long simple root. The negative roots are

$$\bar{\mathfrak{n}} = \mathrm{span}\left\{\sqrt{2}E_{\sigma(i),i}, E_{j,i} - E_{\sigma(i),\sigma(j)}, E_{\sigma(j),i} + E_{\sigma(i),j} \,|\, 1 \le i < j \le n\right\}$$

It is important to note that if $X \in \mathfrak{n}$, then $\theta(X) \in \bar{\mathfrak{n}}$. Further, if $X \in \mathfrak{p}$, then $X + \theta(X) \in \mathfrak{k}$.

To the above Lie algebra decomposition corresponds a minimal parabolic $P_{\mathbb{R}} = M_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}$, where the decomposition is the Langlands decomposition. Later it will become important to know that $M_{\mathbb{R}} = \mathrm{pr}^{-1}(\mathrm{diag}(\eta_1, \eta_2, \ldots, \eta_n, \eta_n, \ldots, \eta_2, \eta_1))$, with $\eta_i \in \{\pm 1\}$

2.2 Quasispherical Representations

Ciubotaru and Trapa developed a Lefschetz functor for spherical representations of classical linear groups. Since spherical representations of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ factor through $\mathrm{Sp}(2n,\mathbb{R})$, these are already understood by the work of Ciubotaru and Trapa; the goal instead is to study the simplest possible representations which do not factor through $\mathrm{Sp}(2n,\mathbb{R})$.

Definition 6 *A representation (ϕ, V) of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ is* genuine *if ζ acts nontrivially on V, where ζ is the nontrivial element of the kernel of* pr.

That is to say, a representation of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ is genuine if the action does not factor through $\mathrm{Sp}(2n,\mathbb{R})$. Genuine representations of subgroups of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ are defined the same way. An important family of genuine representations of $\widetilde{K}_{\mathbb{R}}$ are the square root of the determinant representations, defined by

$$\widetilde{\delta}_k := \det^{\frac{k}{2}}([g,z]) = z^k \tag{2.2}$$

These $\widetilde{K}_{\mathbb{R}}$-types suggest a family of categories of genuine $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ representations, given by $\mathcal{HC}_{\widetilde{\delta}_k}(\lambda, \widetilde{G}_{\mathbb{R}})$. The $\widetilde{K}_{\mathbb{R}}$-type $\widetilde{\delta}_1$ will be abbreviated $\widetilde{\delta}$.

The category of genuine representations most closely analogous to the spherical representations are the so-called *quasi-spherical* representations. These are the objects of the category $\mathcal{HC}_{\widetilde{\delta}_{\pm 1}}(\lambda, \widetilde{G}_{\mathbb{R}})$. The category is the same for either choice of $\widetilde{K}_{\mathbb{R}}$-type.

The goal of this dissertation is to develop a Lefschetz functor for the quasi-spherical representations of the Metaplectic group. In fact, the main theorem will provide a family of functors, corresponding to the family of categories $\mathcal{HC}_{\widetilde{\delta}_k}(\lambda, \widetilde{G}_{\mathbb{R}})$. The focus of this dissertation, however, will be on the case $k = \pm 1$.

2.2.1 Iwahori-Hecke Algebras

Suppose Γ is a linear algebraic group over $\mathbb{Q}_p$, and $\Gamma(F_p)$ is the corresponding group over the residue field of $\mathbb{Q}_p$. There is a canonical map $\phi : \Gamma \to \Gamma(F_p)$. Fix a Borel subgroup B of $\Gamma(\mathbb{Q}_p)$; then $I = \phi^{-1}(B)$ is an Iwahori subgroup of Γ. The space of functions

$$C[I\backslash\Gamma/I] = \left\{ f \in C_c[\Gamma] \mid f(kgk') = f(g)\ \forall k, k' \in I \right\}$$

where $C_c[\Gamma]$ is the space of continuous, compactly supported functions from Γ to $\mathbb{C}$, forms an algebra under convolution. This algebra is called an affine Hecke algebra, denoted $\mathcal{H}(\Gamma)$. It is an associative algebra over $\mathbb{C}[q, q^{-1}]$, with generators $\{T_w\}_{w\in W_{\mathbb{R}}}$ and $\{\theta_x\}_{x\in X}$, where X is the weight lattice of Γ. The parameter q is indeterminate. In fact, if q is set to 1, this algebra becomes the group algebra for the affine Weyl group.

This algebra is also equipped with a natural $*$-operator, inherited from the group, which makes it possible to define a notion of a unitary representation. On generators it is given by

$$q^* = q$$

$$T_w^* = T_{w^{-1}}$$
$$\theta_x^* = T_{w_0}\theta_{-x}T_{w_0}^{-1}$$

where w_0 is the long element of the Weyl group.

Suppose (π, V) is an admissable representation of Γ. Since I is compact open, the space of I-fixed vectors V^I is finite dimensional. The algebra $\mathcal{H}(\Gamma)$ acts on V^I by convolution. Suppose $f \in \mathcal{H}(\Gamma), v \in V^I$. Then:

$$(f * v)(x) = \int_\Gamma f(xy^{-1})\pi(y)v\,dy$$

This action is well-defined since v is left I-invariant.

This functorial construction consumes an admissible representation of Γ with non-zero I-invariant subspaces and produces a $\mathcal{H}(\Gamma)$ module. This is called the Borel-Casselman Correspondence, and it is actually a categorical equivalence. The Borel-Casselman correspondence is an extremely powerful tool for studying the representation theory of p-adic groups, since it reduces infinite dimensional representations of an algebraic group to finite dimensional representations of an algebra.

The algebra $\mathcal{H}(\Gamma)$ is still extremely complicated. However, a further reduction is possible, due to Lusztig [13]. The algebra $\mathcal{H}(\Gamma)$ admits a family of graded versions, called graded affine Hecke algebras (see definition 2), parameterized by Weyl group orbits on the maximal torus of Γ. Given a category $\mathrm{mod}_\chi(\mathcal{H}(\Gamma))$ of $\mathcal{H}(\Gamma)$ modules with a fixed central character χ, it is possible to choose an appropriate graded affine Hecke algebra $\mathbb{H}_R(\mathbf{c})$, and reduce the parameterization of $\mathrm{mod}_\chi(\mathcal{H}(\Gamma))$ to the parameterization of $\mathbb{H}_R(\mathbf{c})$ modules. This construction is extremely subtle, and its details will not be included here, but see for instance [2] for a very clear exposition. The graded affine Hecke algebras also have a natural $*$-operator, inherited from the affine Hecke algebra, providing a notion of unitarity; Barbasch and Moy proved [4] that this series of equivalences preserves unitarity.

These two constructions, taken together, reduce much of the representation theory of Γ to the representation theory of a family of relatively well-behaved associative algebras.

2.3 Metaplectic Group

Defining a Lefschetz functor in the metaplectic case is somewhat more complicated than the linear case. By choosing an Iwahori subgroup $\widetilde{I} \subset \widetilde{G}_{\mathbb{Q}_p}$ the Iwahori-Hecke algebra $\mathcal{H}(\widetilde{G}_{\mathbb{Q}_p})$ can still be defined in the same way as the linear case. The Borel-Casselman correspondence provides a categorical equivalence between representations of $\widetilde{\mathcal{H}}(\widetilde{G}_{\mathbb{Q}_p})$ and representations of $\widetilde{G}_{\mathbb{Q}_p}$ generated by their $\widetilde{I}$-fixed vectors. Unfortunately the algebra $\mathcal{H}(\widetilde{G}_{\mathbb{Q}_p})$ is even more complicated than before.

Fortunately Savin proved that $\mathcal{H}(\widetilde{G}_{\mathbb{Q}_p}) \cong \mathcal{H}(\mathrm{SO}(2n+1))$, the Iwahori-Hecke algebra for the special orthogonal group, and that this isomorphism respects the $*$-operation [12]. Since this group is linear, the work of Lusztig allows the representation theory of $\mathcal{H}(\mathrm{SO}(2n+1))$ to be reduced to the representation theory of a graded affine Hecke algebra. It is important to note, though, that since $\mathrm{SO}(2n+1)$ is a group of type B_n, the underlying root system for the graded affine Hecke algebra will be B_n, rather than C_n.

The correct choice of graded affine Hecke algebra depends on the minimal $\widetilde{K}_{\mathbb{R}}$-type of the $\widetilde{G}_{\mathbb{R}}$. For a quasi-spherical representation of $\widetilde{G}_{\mathbb{R}}$, the correct choice is $\mathbb{H}_{B_n}(\mathbf{c} \equiv 1)$. The functor can be constructed more generally; for minimal $\widetilde{K}_{\mathbb{R}}$-type $\det^{k/2}$ the choice is

$$\mathbf{c}(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ long} \\ k & \text{if } \alpha \text{ short} \end{cases} \tag{2.3}$$

The standard presentation of B_n has roots $\{\pm e_i \pm e_j\}_{1\le i,j\le n} \cup \{e_i\}_{1\le i\le n}$, and simple roots $\{e_i - e_{i+1}\}_{1\le i\le n-1} \cup \{e_n\}$. Denote the reflection through $e_i - e_{i+1}$ by $s_{i,i+1}$; denote the reflection through e_i by $\bar{s}_i$. The Weyl group for B_n is $S_n \ltimes \left(\mathbb{Z}/2\mathbb{Z}\right)^n$. The subgroup S_n is generated by $\{s_{i,i+1}\}$, while $\left(\mathbb{Z}/2\mathbb{Z}\right)^n$ is generated by $\{\bar{s}_i\}$. Explicitly, then, the relations in definition 2 become:

$$s_{i,i+1}\epsilon_i - \epsilon_{i+1}s_{i,i+1} = 1 \tag{2.4}$$

$$s_{i,i+1}\epsilon_j - \epsilon_j s_{i,i+1} = 0 \qquad \text{if } j \neq i, i+1 \tag{2.5}$$

$$\bar{s}_i\epsilon_j - \epsilon_j\bar{s}_i = 0 \qquad \text{if } j \neq i \tag{2.6}$$

$$\bar{s}_i\epsilon_i + \epsilon_i\bar{s}_i = k \tag{2.7}$$

In this dissertation I construct a family of functors $\widetilde{\mathcal{F}}_\delta$, analogous to the Lefschetz functors for linear groups, from the category $\mathcal{HC}_\delta(\lambda)$ to the category of $\mathbb{H}_{\mathrm{B}_n}(\mathbf{c})$ modules with central character λ.

Theorem 7 *Fix $\delta = \det^{k/2}$, a character of $K_{\mathbb{R}}$. For any object X in the category $\mathcal{HC}_\delta(\lambda)$ there is a natural action of $\mathbb{H}_{\mathrm{B}_n}(\mathbf{c})$ on the space $\mathrm{Hom}_{K_{\mathbb{R}}}\Big(\det^{k/2}, (\det^{-1} \otimes X) \otimes \left(\mathbb{C}^{2n}\right)^{\otimes n}\Big)$. This defines an exact functor $\widetilde{F}_\delta$*

$$\widetilde{F}_\delta : \mathcal{HC}_\delta(\lambda) \longrightarrow \mathbb{H}_{\mathrm{B}_n}(\mathbf{c})\text{-}mod$$

such that:

1. *$\widetilde{F}_\delta(I_\delta(\lambda)) = \mathbb{I}_1(\lambda)$ and $\widetilde{F}_\delta(J_\delta(\lambda)) = \mathbb{J}_1(\lambda)$*

2. *If X is an irreducible Hermitian (respectively unitary) object in $\mathcal{HC}_\delta(\lambda)$ then $\widetilde{F}_\delta(X)$ is an irreducible Hermitian (respectively unitary) object in $\mathbb{H}_{\mathrm{B}_n}(\mathbf{c})$-mod*

where $\mathbf{c}(\alpha) = 1$ *if* α *is a long root, and* $\mathbf{c}(\alpha) = k$ *if* α *is a long root.*

This provides an embedding of the quasi-spherical unitary dual of $\widetilde{G}_{\mathbb{R}}$ into the spherical unitary dual of $\mathbb{H}_{\mathrm{B}_n}(\mathbf{c})$.

Since the Borel-Casselman correspondence preserves unitarity, we have the following corollary to the main result:

Corollary 8 *Theorem 7 embeds the quasi-spherical unitary dual of* $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ *into the spherical unitary dual of* $\mathrm{SO}(2n+1, \mathbb{Q}_p)$*, and the quasi-spherical unitary dual of* $\widetilde{\mathrm{Sp}}(2n, \mathbb{Q}_p)$*.*

2.3.1 Relationship with the Theta Correspondence

The functor of 7 identifies the quasi-spherical unitary dual of $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ with the unitary dual of a graded affine Hecke algebra of type B_n. There is another possible way to construct such a correspondence, though, which is via the theta correspondence. An obvious question is whether the two methods coincide.

Definition 9 *Suppose* G' *is a real linear group,* B *a Borel subgroup with Langlands decomposition* $B = MAN$*, and* ω *an irreducible representation of* M*. Denote the Lie subalgebra corresponding to* A *by* $\mathfrak{a}$*. Then define*

$$\mathrm{CS}(\omega, G') = \left\{\nu \in \mathfrak{a} \middle| J_\delta(\nu) \text{ is unitary}\right\}$$

Explicitly, Adams et al. proved the following bijection in [1]:

$$\mathrm{CS}(\delta, \widetilde{G}_{\mathbb{R}}) \xrightarrow{\sim} \mathrm{CS}(1, \mathrm{SO}(n+1, n)) \tag{2.8}$$

This bijection is implemented by the theta correspondence. It allows irreducible unitary representations of $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ to be matched up with irreducible unitary representations of $\mathrm{SO}(n+1, n)$; the Lefschetz functors for linear groups can then translate the $\mathrm{SO}(n+1, n)$ representations into a irreducible unitary $\mathbb{H}_{\mathrm{B}_n}(\mathbf{c})$ modules.

An important question, then, is whether or not this correspondence matches up with the correspondence given by the functor $\widetilde{F}_\delta$ of Theorem 7. Another way to pose this question is to notice that the result in equation (2.8) is really a map of tori, and both Ciubotaru's and Trapa's functor $\mathcal{F}_{\mu_0, V, k}$ and the functor $\widetilde{F}_\delta$ induce maps from the torus of the real group to the torus of $\mathbb{H}_{\mathrm{B}_n}(\mathbf{c})$. It suffices, then, to ask if the map of tori induced by $\widetilde{F}_\delta$ is the same as the map of tori induced by $\mathcal{F}_{\mu_0, V, k}$ composed with the map in (2.8).

In fact, the maps are the same, and can be written down explicitly. In standard coordinates, the roots in a system of type C_n are given by $\{\pm e_i \pm e_j\}_{1 \le i \ne j \le n} \cup \{2e_i\}_{1 \le i \le n}$, while the roots in a system of type B_n are given by $\{\pm e_i \pm e_j\}_{1\leq i\neq j\leq n} \cup \{e_i\}_{1\leq i\leq n}$. In these coordinates, the map of tori is simply the identity. In particular, they both map the infinitesimal character of the oscillator representation of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ to the infinitesimal character of the trivial representation of $\mathbb{H}_{\mathrm{B}_n}(\mathbf{c})$.

CHAPTER 3

PROOF OF THE MAIN RESULT

3.1 Outline of the Construction

Given an object X in $\mathbb{H}_{\mathrm{B}_n}(\mathbf{c})$-mod, the vector space $\widetilde{F}_\delta(X)$ is simply

$$\widetilde{F}_\delta(X) = \mathrm{Hom}_{K_\mathbb{R}}\Big(\det{}^{k/2}, (\det{}^{-1} \otimes X) \otimes \left(\mathbb{C}^{2n}\right)^{\otimes n}\Big)$$

The action of the Weyl group is defined on the $\left(\mathbb{C}^{2n}\right)^{\otimes n}$ term. Very roughly, reflections in the long roots interchange copies of $\mathbb{C}^{2n}$, while reflections in the short roots act by a reflection within each copy of $\mathbb{C}^{2n}$. This picture is precisely correct in the case where X is a principal series representation.

$$\begin{aligned}
\widetilde{F}_\delta(I_\delta(\lambda)) &= \mathrm{Hom}_{\widetilde{K}_\mathbb{R}}\Big(\det{}^{k/2}, \Big(\det{}^{-1} \otimes \mathrm{Ind}_{\widetilde{B}_\mathbb{R}}^{\widetilde{G}_\mathbb{R}}\left(\det{}^{k/2} \otimes e^\lambda \otimes 1\right)\Big) \otimes \left(\mathbb{C}^{2n}\right)^{\otimes n}\Big) \\
&\cong \mathrm{Hom}_{\widetilde{K}_\mathbb{R}}\Big(\det{}^{k/2}, \mathrm{Ind}_{\widetilde{B}_\mathbb{R}}^{\widetilde{G}_\mathbb{R}}\Big(\det{}^{k/2} \otimes \det{}^{-1} \otimes e^\lambda \otimes 1 \otimes \left(\mathbb{C}^{2n}\right)^{\otimes n}\Big)\Big) \\
&\cong \mathrm{Hom}_{\widetilde{M}_\mathbb{R}}\Big(\det{}^{k/2}, \det{}^{k/2} \otimes \det{}^{-1} \otimes \left(\mathbb{C}^{2n}\right)^{\otimes n}\Big) \\
&\cong \mathrm{Hom}_{\widetilde{M}_\mathbb{R}}\Big(\det, \left(\mathbb{C}^{2n}\right)^{\otimes n}\Big)
\end{aligned}$$

But since the action of $\widetilde{M}_\mathbb{R}$ factors through $M_\mathbb{R}$ on both sides:

$$\cong \mathrm{Hom}_{M_\mathbb{R}}\Big(\det, \left(\mathbb{C}^{2n}\right)^{\otimes n}\Big) \tag{3.1}$$

For a particular choice of basis $\{e_i\}_{i=1}^{2n}$ of $\mathbb{C}^{2n}$, the subgroup $M_\mathbb{R}$ can be written as a collection of diagonal matrices in the form $M_\mathbb{R} = \{\mathrm{diag}(\eta_1, \ldots, \eta_n, \eta_n, \ldots, \eta_1) | \eta_i \in \{\pm 1\}\}$. A basis of (3.1) is thus given by

$$\left\{ \phi \in \mathrm{Hom}_{M_\mathbb{R}}\Big(\det, \left(\mathbb{C}^{2n}\right)^{\otimes n}\Big) \;\middle|\; \begin{array}{c} \phi(1) = e_{i_1} \otimes \cdots \otimes e_{i_n} \\ i_j \neq i_k, \sigma(i_k)\ \forall j, k \end{array} \right\} \tag{3.2}$$

where $\sigma(i) = 2n + 1 - i$. If we again imagine that reflections in the long roots in B_n interchange copies of $\mathbb{C}^{2n}$, and that reflections in the short roots act by $e_i \mapsto e_{\sigma(i)}$ in individual copies of $\mathbb{C}^{2n}$, then it is clear that this basis is simply the W-orbit of a single element ϕ such that $\phi(1) = e_1 \otimes \cdots \otimes e_n$. In other words, as vector spaces:

$$\mathrm{Hom}_{M_{\mathbb{R}}}\Big(\det, \left(\mathbb{C}^{2n}\right)^{\otimes n}\Big) \cong \mathbb{C}[W] \tag{3.3}$$

Unfortunately there is no natural, functorial way to define this Weyl group action directly on $X \otimes \left(\mathbb{C}^{2n}\right)^{\otimes n}$. For technical reasons the action is only well defined on the space $\widetilde{F}_\delta(X)$. The reflections through long roots are implemented by an operator called Ω_{ij}, while reflections through short roots are implemented by an operator called ξ.

Fix a bilinear, nondegenerate form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, and a basis $\mathbf{B}$ of $\mathfrak{g}$, together with a duality operation $E \mapsto E^*$, such that the $\mathbf{B}$ is self-dual in the sense that

$$\langle E, F\rangle = \begin{cases} 1 & \text{if } F = E^* \\ 0 & \text{otherwise} \end{cases}$$

Define an operator Ω_{ij} on $X \otimes \left(\mathbb{C}^{2n}\right)^{\otimes n}$ by

$$\Omega_{ij} = \sum_{E \in \mathbf{B}} (E)_i \otimes (E^*)_j$$

where $(E)_i$ indicates the action of E on the i^{th} factor of the tensor product, counting from zero. The similarity between this operator and the Casimir operator is not accidental, and is leveraged in many of the proofs. In type A the analogous operator actually acts by interchanging coordinates, but that luck does not hold in other cases; in particular this operator does not square to the identity in the present case.

The reflections through the short root are implemented by multiplication in the copies of $\left(\mathbb{C}^{2n}\right)^{\otimes n}$ by the element $\xi = iJ$. This element implements the Cartan involution θ, in the sense that ξ is an element of the complexification of $G_{\mathbb{R}}$ such that $\xi X \xi^{-1} = \theta(X)$ for all $X \in \mathfrak{g}$.

Lemma 10 *For any object X in $\mathcal{HC}_\delta(\lambda)$, there is a natural action π of $\mathbb{H}_{\mathrm{B}_n}(\mathbf{c})$ on $\widetilde{F}_\delta(X) = \mathrm{Hom}_{K_{\mathbb{R}}}\Big(\det^{k/2}, (\det^{-1} \otimes X) \otimes \left(\mathbb{C}^{2n}\right)^{\otimes n}\Big)$ given by*

$$\begin{aligned} \pi(s_{i,j}) &= -\Omega_{ij} \\ \pi(\bar{s}_i) &= \textit{multiplication by } -\xi \textit{ in the } i^{th} \textit{ coordinate} \\ \pi(\epsilon_i) &= \sum_{j=0}^{i-1} \Omega_{ji} + n \end{aligned}$$

The difficult step in proving this lemma is proving that relation (2.7) holds. In fact, it is this step which fixes the choice of the parameter $\mathbf{c}$ for the Hecke algebra $\mathbb{H}_{\mathrm{B}_n}(\mathbf{c})$. This proof is also one of the major obstructions in generalizing the results, since it relies explicitly on the fact that $\det^{k/2}$ is one-dimensional.

The proof that $\widetilde{F}_\delta$ preserves unitarity and sends irreducible representations to irreducible representations follows formally from a computation similar to the proof of relation (2.7) together with the fact that $\widetilde{F}_\delta(I_\delta(\nu)) = \mathbb{I}_1(\nu)$. The proof of the latter fact follows from (3.3); because $\mathbb{I}_1(\nu)$ is also isomorphic to $\mathbb{C}[W]$ as a vector space, and the action of W on both is the same, it remains only to show that the representation has the correct infinitesimal character. This can be shown by an explicit calculation.

3.2 Proof of Main Theorem

3.2.1 Normalization

To define the Casimir operator, as well as the Hecke algebra action to be described later, a basis B needs to be found which is orthonormal with respect to a carefully chosen nondegenerate symmetric bilinear ad-invariant form. The appropriate form will turn out to be $\frac{1}{2}$ of the trace form, i.e.:

$$\langle X, Y\rangle = \frac{1}{2}\operatorname{tr}(XY)$$

In fact, it will turn out that this normalization is the only choice that results in a well-defined action of $\mathbb{H}_{\mathrm{B}_n}(\mathbf{c})$.

In particular, the basis needs to possess a duality operation $E \mapsto E^*$ mapping the basis to itself, such that

$$\langle X, Y\rangle = \begin{cases} 1 & Y = X^* \\ 0 & \text{o.w.} \end{cases}$$

Such a basis can be generated by appropriately choosing elements of the root spaces for $\mathfrak{a}$. The duality operation carries elements of $\mathfrak{a}$ to themselves, and elements of the root space $\mathfrak{g}_\alpha$ to elements of $\mathfrak{g}_{-\alpha}$.

Then the resulting basis B is as follows:

	E	E^*
$\mathfrak{a}$:	$E_{i,i} - E_{\sigma(i),\sigma(i)}$	self
$\mathfrak{n}, \bar{\mathfrak{n}}$:	$\sqrt{2}E_{i,\sigma(i)}$	$\sqrt{2}E_{\sigma(i),i}$
	$E_{i,j} - E_{\sigma(j),\sigma(i)}$	$E_{j,i} - E_{\sigma(i),\sigma(j)}$
	$E_{i,\sigma(j)} + E_{j,\sigma(i)}$	$E_{\sigma(j),i} + E_{\sigma(i),j}$

3.2.2 $\mathbb{H}_R(\mathbf{c})$ Action

With the choice of basis above, the operator Ω_{ij} on $X \otimes \left(\mathbb{C}^{2n}\right)^{\otimes n}$ is defined by

$$\Omega_{ij} = \sum_{E \in B} (E)_i \otimes (E)_j$$

In the type A_n case, this operator simply acts by reflection. In other cases the action is not quite so simple; in particular, $\Omega_{ij}^2 \neq 1$. The action of Ω_{ij} is encapsulated in the following lemma:

Lemma 11 *On $\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$, $\sum_{E \in B} (E)_i \otimes (E^*)_j$ acts by $R - 2n\,\mathrm{pr}_1$, where R interchanges the coordinates, and pr_1 is the projection onto the $U(\mathfrak{g})$-trivial component.*

Proof of lemma: For the sake of simplicity, the following proof will deal with the action of $\Omega = \Omega_{1,2}$ on $V \otimes V$; the results generalize easily.

The operator Ω is an explicit version of an operator defined in terms of the Casimir element $C \in \mathcal{U}(\mathfrak{g})$. The proof is easier to understand in terms of this more fundamental operator.

The Casimir element C is defined as

$$C = \sum_{E \in B} EE^*$$

where B and E^* are the basis and duality operation defined earlier.

There is an embedding $\Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ given by

$$\Delta(x) = 1 \otimes x + x \otimes 1$$

That implies the following identity

$$\begin{aligned}\Delta(xy) &= (1 \otimes x + x \otimes 1)(1 \otimes y + y \otimes 1) \\ &= x \otimes y + y \otimes x + xy \otimes 1 + 1 \otimes xy\end{aligned}$$

Then the operator Ω can be rewritten as

$$\Omega = \frac{1}{2}(\Delta(C) - 1 \otimes C - C \otimes 1)$$

The Casimir operator is central in $\mathcal{U}(\mathfrak{g})$ and operates by a scalar on each of its irreducible components. In particular, it is possible to compute the action of Ω by computing the action of C on each irreducible component under the action of $\Delta(C)$, $1 \otimes C$, and $C \otimes 1$.

Suppose V has basis $\{e_i\}$ and $\sigma(i) = 2n - i$, as earlier. It is well known that

$$V \otimes V = S^2V \oplus \wedge^2 V$$

Each of these is an invariant subspace under the action of $\mathfrak{gl}(n, \mathbb{C})$. The operator R, which interchanges the two coordinates, acts on S^2V by 1 and $\wedge^2 V$ by -1. Explicitly, $S^2V = \operatorname{span}\langle e_i \otimes e_j + e_j \otimes e_i\rangle$ and $\wedge^2 V = \operatorname{span}\langle e_i \otimes e_j - e_j \otimes e_i,\, i \neq j\rangle$.

In $\mathfrak{sp}(2n, \mathbb{R})$, because of equation 2.1, $\wedge^2 V$ contains a copy of the trivial representation. It is the span of the vector $\sum_{i \leq n} e_i \otimes e_{\sigma(i)} - e_{\sigma(i)} \otimes e_i$; $\Delta(C)$ acts on this by 0. Using the basis given in the table in section 3.2.1, it is straightforward to show, by computation on the highest weight vectors, the following facts:

1. $C \cdot v = (2n + 1)v,\ \forall v \in V$
2. $\Delta(C)v = 2 + 2(2n + 1)v\, \forall v \in S^2V$
3. $\Delta(C)v = -2 + 2(2n + 1)v\, \forall v \in {}^{\wedge^2 V}\!/_{1}$
4. $\Delta(C)v = 0$ if v is a multiple of $\sum_{i \leq n} e_i \otimes e_{\sigma(i)} - e_{\sigma(i)} \otimes e_i$

Collecting terms, Ω acts by:

$$\begin{aligned}
\pi(\Omega) &= \frac{1}{2}(\Delta(C) - 1 \otimes C - C \otimes 1) \\
&= \frac{1}{2}(2 + 2(2n + 1)) \operatorname{pr}_{S^2V} + \frac{1}{2}(-2 + 2(2n + 1)) \operatorname{pr}_{\wedge^2 V/1} - (2n + 1) \\
&= \operatorname{pr}_{S^2V} - \operatorname{pr}_{\wedge^2 V/1} + (2n + 1)(\operatorname{pr}_{S^2V} + \operatorname{pr}_{\wedge^2 V/1}) - (2n + 1) \\
&= \operatorname{pr}_{S^2V} - \operatorname{pr}_{\wedge^2 V/1} + (2n + 1) - (2n + 1) \operatorname{pr}_1 - (2n + 1) \\
&= \operatorname{pr}_{S^2V} - \operatorname{pr}_{\wedge^2 V/1} - (2n + 1) \operatorname{pr}_1
\end{aligned}$$

The operator R can be re-written as $R = \operatorname{pr}_{S^2V} - \operatorname{pr}_{\wedge^2 V}$, so the above equation can be rephrased as:

$$\begin{aligned}
\pi(\Omega) &= R + \operatorname{pr}_1 - (2n + 1) \operatorname{pr}_1 \\
&= R - 2n \operatorname{pr}_1
\end{aligned}$$

$\square$

Although $\Omega_{ij}^2 \neq 1$ on $X \otimes \left(\mathbb{C}^{2n}\right)^{\otimes n}$, it does square to the identity on $\widetilde{F}_\delta(X)$. In fact, the following lemma holds:

Lemma 12 *For any object X in $\mathcal{HC}_\delta(\lambda)$, there is a natural action π of $\mathbb{H}_{\mathrm{B}_n}(\mathbf{c})$ on* $\mathrm{Hom}_{K_{\mathbb{R}}}\Big(\det^{\frac{k}{2}}, \big(\det^{-1}\otimes X\big)\otimes\big(\mathbb{C}^{2n}\big)^{\otimes n}\Big)$ *given by*

$$\begin{aligned}\pi(s_{i,j}) &= -\Omega_{ij}\\ \pi(\bar{s}_i) &= \textit{multiplication by } -\xi \textit{ in the } i^{th} \textit{ coordinate}\\ \pi(\epsilon_i) &= \sum_{j=0}^{i-1}\Omega_{ji}+n\end{aligned}$$

Proof of lemma: First, note that the operator Ω_{ij} does, in fact, preserve the space $\widetilde{F}_\delta(X)$. This is because it commutes with the action of K. Observe that if $k\in K$:

$$\begin{aligned}k\cdot\Omega_{i,j}\cdot k^{-1}\cdot v &= k\cdot\sum_{E\in B}(E)_i\otimes(E^*)_j\cdot k^{-1}\cdot v\\ &= \sum_{E\in B}(kEk^{-1})_i\otimes(kE^*k^{-1})_j v\end{aligned}$$

But conjugation by K preserves the trace form. Thus the set kBk^{-1} is an orthonormal, self-dual basis; since the Casimir operator is independent of choice of basis, the term in the second line is actually equal to $\Omega_{i,j}$.

There are two more steps. First, it is necessary to show that this is a $\mathbb{C}[W_{\mathbb{R}}]$ representation, then the commutation relations for $\mathbb{H}_{\mathrm{B}_n}(\mathbf{c})$ must be proved.

1. The $\mathbb{C}[W_{\mathbb{R}}]$ Action

   The group $W_{\mathbb{R}}$ is generated by the reflections $s_{i,i+1}$ and $\bar{s}_n$ together with the relations

$$\begin{aligned}(s_{i,i+1}s_{j,j+1})^2 &= 1 && j\neq i\pm 1\\ (s_{i,i+1}\bar{s}_n)^2 &= 1 && \text{if } i+1\neq n\\ (s_{i,i+1}s_{i+1,i+2})^3 &= 1\\ (s_{n-1,n}\bar{s}_n)^4 &= 1\end{aligned}$$

   The first two relations simply state the the operators commute; this is clear from the definition of the action, since those operators act on different factors in the representation. The third relation, and the fact that $s_{i,i+1}$ has order two, simply comes from the fact that the $s_{i,i+1}$ act on by transposition of coordinates on $(\mu_0^*\otimes(\mathbb{C}^{2n})^{\otimes n})^{M_{\mathbb{R}}}$. This gives an action of $S_n\subset W_{\mathbb{R}}$; the third relation is simply the braid relation in this group.

   The last relation is easily checked by explicit computation on the basis given in (3.2).

2. $S(\mathfrak{a})$ Action

First it is necessary to verify that

$$\pi(s_{i,i+1})\pi(\epsilon_i) - \pi(\epsilon_{i+1})\pi(s_{i,i+1}) = 1$$

This is easy to show from the definition of the action of ϵ_i:

$$\pi(s_{i,i+1})\sum_{j<i}\Omega_{j,i} - \sum_{j<i+1}\Omega_{j,i+1}\pi(s_{i,i+1}) = 1$$

But $s_{i,i+1}$ simply interchanges coordinates:

$$\begin{aligned}\pi(s_{i,i+1})\sum_{j<i}\Omega_{j,i} - \pi(s_{i,i+1})\sum_{j<i}\Omega_{j,i} + -\Omega_{i,i+1}s_{i,i+1} &= 1\\ s_{i,i+1}s_{i,i+1} &= 1\end{aligned}$$

It is also necessary to prove that the ϵ_i commute. Suppose $l < m$. Most of the terms commute trivially:

$$\begin{aligned}[\epsilon_l, \epsilon_m] &= \left[\sum_{i<l}\Omega_{i,l}, \sum_{j<m}\Omega_{j,m}\right]\\ &= \sum_{i<l}[\Omega_{i,l}, \Omega_{i,m} + \Omega_{l,m}]\end{aligned}$$

The terms within the sum are actually each individually equal to zero:

$$\begin{aligned}&\Bigg[\sum_{E\in B}(E)_i\otimes(E^*)_l\otimes(1)_m, \sum_{F\in B}(F)_i\otimes(1)_l\otimes(F^*)_m\\ &\qquad\qquad + \sum_{F\in B}(1)_i\otimes(F)_l\otimes(F^*)_m\Bigg]\\ &= \sum_{F\in B}\Bigg[\sum_{E\in B}(E)_i\otimes(E^*)_l, (F)_i\otimes(1)_l\otimes(F^*)_m\\ &\qquad\qquad + (1)_i\otimes(F)_l\otimes(F^*)_m\Bigg]\\ &= \sum_{F\in B}[\Omega_{i,l}, \Delta_{i,l}(F)]\otimes(F^*)_m\end{aligned}$$

Since the Casimir operator is central in the universal enveloping algebra, the commutator in the last line is equal to zero.

The most difficult identity to prove is that

$$\pi(\bar{s}_n)\pi(\epsilon_n) + \pi(\epsilon_n)\pi(\bar{s}_n) = 1$$

Begin by computing $\pi(\bar{s}_j)\pi(\Omega_{i,j})$:

$$\begin{aligned}
\pi(\bar{s}_j)\pi(\Omega_{i,j}) &= -\pi\left(\sum_{E\in B}(E)_i \otimes (\xi E^*)_j\right) \\
&= -\pi\left(\sum_{E\in B\cap\mathfrak{k}}(E)_i \otimes (E^*\xi)_j - \sum_{E\in B\cap\mathfrak{p}}(E)_i \otimes (E^*\xi)_j\right) \\
&= \pi\left(\sum_{E\in B\cap\mathfrak{k}}(E)_i \otimes (E^*)_j - \sum_{E\in B\cap\mathfrak{p}}(E)_i \otimes (E^*)_j\right)\pi(\bar{s}_j)
\end{aligned}$$

This is true because $\mathfrak{p}$ and $\mathfrak{k}$ are the ± 1 eigenspaces under conjugation by ξ. The sums here will come up again; they will be denoted by

$$\begin{aligned}
\Omega^{\mathfrak{k}}_{i,j} &= \sum_{E\in B\cap\mathfrak{k}}(E)_i \otimes (E^*)_j \\
\Omega^{\mathfrak{p}}_{i,j} &= \sum_{E\in B\cap\mathfrak{p}}(E)_i \otimes (E^*)_j
\end{aligned}$$

Now we have

$$\pi(\bar{s}_j)\pi(\Omega_{i,j}) = \pi\left(\Omega^{\mathfrak{k}}_{i,j} - \Omega^{\mathfrak{p}}_{i,j}\right)\pi(\bar{s}_j)$$

That implies the following formula

$$\pi(\bar{s}_j)\pi(\Omega_{i,j}) + \pi(\Omega_{i,j})\pi(\bar{s}_j) = 2\pi(\bar{s}_j)\pi(\Omega^{\mathfrak{k}}_{i,j})$$

Summing over i and setting $j = n$, this yields

$$\pi(\bar{s}_n)\pi(\epsilon_n - r) + \pi(\epsilon_n - r)\pi(\bar{s}_n) = 2\pi(\bar{s}_n)\pi\left(\sum_{i=0}^{n-1}\Omega^{\mathfrak{k}}_{i,n}\right)$$

In the specific case of $k = n$, the operator $\sum_{i=0}^{k-1}\sum_{E\in B\cap\mathfrak{k}}(E^*)_i \otimes (E)_k$ can be simplified to $\sum_{E\in B\cap\mathfrak{k}}(E^*)_k E - (E^*E)_k$, since E is acting on every factor of $(\mathbb{C}^{2n})^{\otimes n}$ except the last one (which is the reason for the correction factor $\sum(E^*E)_k$). The correction factor, in fact, is merely the Casimir element of $\mathcal{U}(\mathfrak{k})$, acting on the k^{th} factor.

Then the equation above becomes

$$\begin{aligned}
&\pi(\bar{s}_n)\pi(\epsilon_n - r) + \pi(\epsilon_n - r)\pi(\bar{s}_n) \\
&\qquad = 2\pi(\bar{s}_n)\pi\left(\sum_{E\in B\cap\mathfrak{k}}(E^*)_n E - \sum_{E\in B\cap\mathfrak{k}}(E^*E)_n\right)
\end{aligned}$$

The Casimir operator in $\mathfrak{k}$ acts on V by n. Since $E \in \mathfrak{k}$, it acts by $\mu = \det^{k/2}$; then we have

$$\pi(\bar{s}_n)\pi(\epsilon_n - r) + \pi(\epsilon_n - r)\pi(\bar{s}_n)$$
$$= 2\pi(\bar{s}_n)\pi\left(\sum_{E \in B \cap \mathfrak{k}} (E^*)_n \mu(E) - n\right)$$

Since $\mu = \det^{k/2}$, it is one-dimensional, and $\mu(E) = 0$ unless $E \in \mathfrak{z}$, the center of $\mathfrak{k}$.

Since $\mathfrak{k}$ is the lie algebra of $\mathrm{U}(n)$ regarded as a real group, the center of $\mathfrak{k}$ is one-dimensional and corresponds to $\mathrm{U}(1)$. The element $\sum_{l \leq n} E_{l,\sigma(l)} - E_{\sigma(l),l}$ generates it. Since $\det^{k/2}\left(\sum_{l \leq n} E_{l,\sigma(l)} - E_{\sigma(l),l}\right) = \frac{ik}{2}$, the formula above becomes

$$\begin{aligned}
&\pi(\bar{s}_n)\pi(\epsilon_n - r) + \pi(\epsilon_n - r)\pi(\bar{s}_n) \\
&\qquad = 2\pi(\bar{s}_n)\pi\Big(\frac{ik}{2}\sum_{l \leq n}(E_{\sigma(l),l} - E_{l,\sigma(l)}) - n\Big) \\
&\qquad = 2\pi(\bar{s}_n)\pi\Big(-\frac{k}{2}\xi - n\Big) \\
&\qquad = k\pi(\bar{s}_n)\pi(\bar{s}_n) - 2n\pi(\bar{s}_n) \\
&\pi(\bar{s}_n)\pi(\epsilon_n) + \pi(\epsilon_n)\pi(\bar{s}_n) = k - 2n\pi(\bar{s}_n) + 2r\pi(\bar{s}_n)
\end{aligned}$$

The commutation relation holds, then, if $r = n$.

□

The following corollary follows from the fact that the K and H actions commute.

Corollary 13 *The functor $\widetilde{F}_\delta$ is exact.*

3.3 Explicit Computation of the H-Action

When X is a principal series representation, explicit computation of the action of $\mathbb{H}_{\mathrm{B}_n}(\mathbf{c})$ on $\widetilde{F}_\delta(X)$ is a tractable problem. In fact, since the action of $\Omega_{i,j}$ is already understood, it merely remains to compute the action of $\Omega_{0,l}$, $1 \leq l \leq n$. The computations in this section follow the outline of the type A_n computations in [6].

A standard module for $\mathbb{H}_{\mathrm{B}_n}(\mathbf{c})$ is generated by a single $W_{\mathbb{R}}$-cyclic vector, which is an eigenvector for the $\{\epsilon_i\}$. It remains simply to find the vector, and prove that it holds the required properties. In fact, then, it suffices to compute the action of $\Omega_{0,l}$ on this particular vector.

The vector $v = 1 \otimes e_1 \otimes \ldots \otimes e_n$ is $W_{\mathbb{R}}$-cyclic in $(\det^* \otimes (\mathbb{C}^{2n})^{\otimes n})^{M_{\mathbb{R}}}$; this is relatively clear from the definition of the $\mathbb{C}[W_{\mathbb{R}}]$-action. The appropriate vector in $\widetilde{F}_\delta(X)$ is the one corresponding to this under Frobenius reciprocity.

The space $\widetilde{F}_\delta(X)$ is isomorphic to the space of vectors in

$$\mathrm{Ind}_{P_{\mathbb{R}}\times G_{\mathbb{R}}\times\cdots\times G_{\mathbb{R}}}^{G_{\mathbb{R}}^{n+1}}(((\det{}^{k/2}\otimes\det{}^*)\otimes e^\nu\otimes 1)\otimes(\mathbb{C}^{2n})^{\otimes n})$$

which transform like $\det^{\frac{k}{2}}$ under the diagonal action of $K_{\mathbb{R}}$, or alternatively the vectors in

$$\mathrm{Ind}_{P_{\mathbb{R}}\times G_{\mathbb{R}}\times\cdots\times G_{\mathbb{R}}}^{G_{\mathbb{R}}^{n+1}}((\det{}^*\otimes e^\nu\otimes 1)\otimes(\mathbb{C}^{2n})^{\otimes n}) \tag{3.4}$$

which transform trivially. The first stage of defining the $W_{\mathbb{R}}$-cyclic vector is as an element of the representation

$$\mathrm{Ind}_{P_{\mathbb{R}}}^{G_{\mathbb{R}}}((\det{}^*\otimes e^\nu\otimes 1)\otimes(\mathbb{C}^{2n})^{\otimes n})$$

We have that $G_{\mathbb{R}}=K_{\mathbb{R}}M_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}$; then define f^Δ as an element of the above space such that

$$f^\Delta(kman)=(man)^{-1}v$$

This vector is $K_{\mathbb{R}}$ invariant. Then define an element of the space in above in equation 3.4 by

$$f(x_0,x_1,\ldots,x_n)=\pi_1(x_1^{-1}x_0)\ldots\pi_n(x_n^{-1}x_0)f^\Delta(x_0)$$

with π_i denoting the action in the $(i+1)^{st}$ factor.

The computation of the $\Omega_{i,j}$ action can be simplified by decomposing it into three cases: $E\in\mathfrak{n},\bar{\mathfrak{n}}$, or $\mathfrak{a}$.

Lemma 14 *Assume $E\in\mathfrak{a}+\mathfrak{n}$, $F\in\mathfrak{g}$. Then*

$$(((E)_0\otimes(F)_l)f)(1)=\pi_0(E)\pi_l(F)v$$

If $E\in\mathfrak{n}$, then

$$(((E)_0\otimes(F)_l)f)(1)=0$$

Proof of lemma:

$$(((E)_0\otimes(F)_l)f)(1)=\left.\frac{d^2}{duds}\right|_{u=s=0}f(e^{-uE},e^{-\delta_{l,1}sF},\ldots,e^{-\delta_{l,n}sF})$$

That is, e^{-uH} in the first coordinate, e^{-sF} in the $(l+1)^{st}$ coordinate, and 1 elsewhere.

$$\begin{aligned}(((E)_0\otimes(F)_l)f)(1)&=\left.\frac{d^2}{duds}\right|_{u=s=0}\pi_l(e^{sF}e^{-uE})\prod_{\substack{l'\neq l\\ l'>0}}\pi_{l'}(e^{-uE})\;f^\Delta(e^{-uE})\\&=\left.\frac{d^2}{duds}\right|_{u=s=0}\pi_l(e^{sF}e^{-uE})\prod_{\substack{l'\neq l\\ l'>0}}\pi_{l'}(e^{-uE})\prod_{0\le l'\le l}\pi_{l'}(e^{uE})v\end{aligned}$$

$$= \left.\frac{d^2}{duds}\right|_{u=s=0} \pi_l(e^{sF})\pi_0(e^{uE})v$$
$$= \pi_l(F)\pi_0(E)v$$

The second line uses the fact that $E \in \mathfrak{a} + \mathfrak{n}$.

Further, notice that if $E \in \mathfrak{n}$, then $e^{-uE} \in N_{\mathbb{R}}$ which acts trivially on f. Then the derivative of the action is zero.

□

Lemma 15 *Suppose* $E \in \bar{\mathfrak{n}}$*. Then*

$$((E)_0 \otimes (E^*)_l)f(1) = \left(\pi_l(-E^*E) + \pi_l(E^*) \sum_{l' \neq l, l' > 0} \pi_{l'}(-E - \theta(E)) \right) v$$

Proof of lemma: Since $E \in \bar{\mathfrak{n}}$, $E + \theta(E) \in \mathfrak{k}$. Define $H = E + \theta(E)$. Since $\theta(E) \in \mathfrak{n}$, using lemma 14 we see that

$$((H)_0 \otimes (E^*)_l)f = ((E)_0 \otimes (E^*)_l)f$$

Following the computation in the lemma 14 we see that

$$\begin{aligned}
((H)_0 \otimes (E^*)_l)f(1) &= \left.\frac{d^2}{duds}\right|_{u=s=0} \pi_l(e^{sE^*}e^{-uH}) \prod_{l' \neq l, l' > 0} \pi_{l'}(e^{-uH})\; f^{\Delta}(e^{-uH}) \\
&= \left.\frac{d^2}{duds}\right|_{u=s=0} \pi_l(e^{sE^*}e^{-uH}) \prod_{l' \neq l, l' > 0} \pi_{l'}(e^{-uH})\; f^{\Delta}(1) \\
&= \left.\frac{d}{du}\right|_{u=0} \pi_l(E^*)\pi_l(e^{-uH}) \prod_{l' \neq l, l' > 0} \pi_{l'}(e^{-uH})\; v \\
&= \pi_l(E^*) \left(\pi_l(-H) + \sum_{l' \neq l, l' > 0} \pi_{l'}(-H) \right) v \\
&= \pi_l(E^*) \left(\pi_l(-E - \theta(E)) + \sum_{l' \neq l, l' > 0} \pi_{l'}(-E - \theta(E)) \right) v \\
&= \left(\pi_l(-E^*E) + \pi_l(E^*) \sum_{l' \neq l, l' > 0} \pi_{l'}(-E - \theta(E)) \right) v
\end{aligned}$$

The second line uses the fact that $e^{-uH} \in K_{\mathbb{R}}$, and f^{Δ} is $K_{\mathbb{R}}$-invariant.

□

Along with the basis table given earlier, we now have enough information to compute the actions of $\Omega_{0,l}$.

We have that

$$\begin{aligned}\Omega_{0,l} &= \sum_{E\in B}(E)_0\otimes(E^*)_l\\ &= \sum_{E\in B\cap\mathfrak{n}}(E)_0\otimes(E^*)_l+\sum_{E\in B\cap\mathfrak{a}}(E)_0\otimes(E^*)_l+\sum_{E\in B\cap\bar{\mathfrak{n}}}(E)_0\otimes(E^*)_l\\ &= \sum_{E\in B\cap\mathfrak{a}}(E)_0\otimes(E^*)_l+\sum_{E\in B\cap\bar{\mathfrak{n}}}(E)_0\otimes(E^*)_l\end{aligned}$$

The remainder of the computation will be broken down into the $\mathfrak{a}$ and $\bar{\mathfrak{n}}$ cases. First, if $E\in\mathfrak{a}$, then $E=E_{i,i}-E_{\sigma(i),\sigma(i)}$:

$$\begin{aligned}&((E_{i,i}-E_{\sigma(i),\sigma(i)})_0\otimes(E_{i,i}-E_{\sigma(i),\sigma(i)})_l)f(1)\\ &\qquad=\pi_0(E_{i,i}-E_{\sigma(i),\sigma(i)})\pi_l(E_{i,i}-E_{\sigma(i),\sigma(i)})v\end{aligned}$$

Note that $\rho(E_{i,i}-E_{\sigma(i),\sigma(i)})=n-i+1$; the action is zero except when $i=l$, so:

$$\begin{aligned}\left(\sum_{E\in B\cap\mathfrak{a}}(E)_0\otimes(E^*)_1\right)f(1) &= (\nu+\rho)(E_{l,l}-E_{\sigma(l),\sigma(l)})v\\ &= (\nu(E_{l,l}-E_{\sigma(l),\sigma(l)})+n-l+1)v\end{aligned}$$

For $E\in\bar{\mathfrak{n}}$ we have:

$$\begin{aligned}&((E_{j,i}-E_{\sigma(i),\sigma(j)})_0\otimes(E_{i,j}-E_{\sigma(j),\sigma(i)})_l)f(1)\\ &\quad=\Bigg[-\pi_l(E_{i,i}+E_{\sigma(j),\sigma(j)})\\ &\qquad+\pi_l(E_{i,j}-E_{\sigma(j),\sigma(i)})\sum_{l'\neq l,l'>0}\pi_{l'}(-E_{j,i}+E_{\sigma(i),\sigma(j)}\\ &\qquad\qquad+E_{i,j}-E_{\sigma(j),\sigma(i)})\Bigg]v\end{aligned}$$

$$\begin{aligned}&((E_{\sigma(j),i}+E_{\sigma(i),j})_0\otimes(E_{i,\sigma(j)}+E_{j,\sigma(i)})_l)f(1)\\ &\quad=\Bigg[-\pi_l(E_{i,i}+E_{j,j})\\ &\qquad+\pi_l(E_{i,\sigma(j)}+E_{j,\sigma(i)})\sum_{l'\neq l,l'>0}\pi_{l'}(E_{-\sigma(j),i}-E_{\sigma(i),j}\\ &\qquad\qquad+E_{i,\sigma(j)}+E_{j,\sigma(i)})\Bigg]v\end{aligned}$$

$$((\sqrt{2}E_{\sigma(i),i})_0 \otimes (\sqrt{2}E_{i,\sigma(i)})_l)f(1)$$
$$= 2\left[-\pi_l(E_{i,i}) + \pi_l(E_{i,\sigma(i)}) \sum_{l'\neq l, l'>0} \pi_{l'}\left(-E_{\sigma(i),i} - E_{i,\sigma(i)}\right)\right] v$$

Because $v = 1 \otimes e_1 \otimes e_2 \otimes \ldots \otimes e_n$, the action of $\Omega_{0,l}$ can be computed quite explicity. When $E \in \mathfrak{a}$, the action is non-zero exactly when $i = l$, ie

$$\left(\sum_{E\in B\cap\mathfrak{a}} (E)_0 \otimes (E)_l\right) f(1) = \big(\nu(E_{i,i} - E_{\sigma(i),\sigma(i)}) + n + 1 - l\big)\, v$$

When $E \in \bar{\mathfrak{n}}$, it is easiest to break up the sum as

$$\begin{aligned}\left(\sum_{E\in B\cap\bar{\mathfrak{n}}} (E)_0 \otimes (E)_l\right) f(1) \\ = \Bigg(\sum_{i>j}(E_{j,i} - E_{\sigma(i),\sigma(j)})_0 \otimes (E_{i,j} - E_{\sigma(j),\sigma(i)})_l \\ + \sum_{i>j}(E_{\sigma(j),i} + E_{\sigma(i),j})_0 \otimes (E_{i,\sigma(j)} + E_{j,\sigma(i)})_l \\ + 2\sum_i (E_{\sigma(i),i})_0 \otimes (E_{i,\sigma(i)})_l\Bigg) v\end{aligned}$$

Proceeding sum by sum, note that in the first case, the action is zero unless $i = l$ or $j = l$. When $i = l$, the action reduces simply to

$$((E_{j,i} - E_{\sigma(i),\sigma(j)})_0 \otimes (E_{i,j} - E_{\sigma(j),\sigma(i)})_i)f(1) = -\pi_l(E_{l,l})v$$

or when $j = l$

$$= \pi_l(E_{i,l})\pi_i(E_{l,i})v$$

The first sum thus evaluates to:

$$\sum_{i<j}((E_{j,i} - E_{\sigma(i),\sigma(j)})_0 \otimes (E_{i,j} - E_{\sigma(j),\sigma(i)})_j)f(1) = \left[\sum_{i<l} \Omega_{i,l} - (n-l)\right] v$$

The second sum is again zero except when $i = l$ or $j = l$. In either case it reduces to

$$\begin{aligned}\sum_{i<j}((E_{\sigma(j),i} + E_{\sigma(i),j})_0 \otimes (E_{i,\sigma(j)} + E_{j,\sigma(i)})_l)f(1) &= -\sum_{i\neq l} \pi_l(E_{l,l})v \\ &= -(n-1)v\end{aligned}$$

The final sum is zero unless $i = l$, which yields a final summation

$$\sum_i ((\sqrt{2}E_{\sigma(i),i})_0 \otimes (\sqrt{2}E_{i,\sigma(i)})_l) f(1) = -2v$$

Collecting terms, the total action is

$$\begin{aligned}
\Omega_{0,l}\, f(1) &= \sum_{E \in B \cap (\mathfrak{a} \cup \bar{\mathfrak{n}})} (E)_0 \otimes (E^*)_l\, f(1) \\
&= \left[\nu(E_{i,i} - E_{\sigma(i),\sigma(i)}) + n + 1 - l - \sum_{i<l} \Omega_{i,l} - (n-l) - (n-1) - 2 \right] v \\
&= \left[\nu(E_{i,i} - E_{\sigma(i),\sigma(i)}) - \sum_{i<l} \Omega_{i,l} - n \right] v
\end{aligned}$$

From the definition of the $S(\mathfrak{a})$ action, we have

$$\begin{aligned}
(\Theta(\epsilon_l) f)(1) &= ((\textstyle\sum_{i<l} \Omega_{0,i} + r) f)(1) \\
&= (\nu(E_{i,i} - E_{\sigma(i),\sigma(i)}) - n + r) v
\end{aligned}$$

We see that if we choose $r = n$, the infinitesimal character of $\widetilde{F}_\delta(X)$ as a module of the Hecke algebra is the same as the infinitesimal character of X.

3.4 Preservation of Unitarity and Irreducibility

3.4.1 Invariant Hermitian Forms

Definition 16 *Suppose $G_{\mathbb{R}}$ is a real reductive group and X is a $(\mathfrak{g}, K)$ module. A Hermitian form $\langle \cdot, \cdot \rangle$ is invariant if*

1. $\langle E \cdot x, y \rangle = -\langle x, \bar{E} \cdot y \rangle$, $\forall x, y \in X$, $E \in \mathfrak{g}$ *where $\bar{E}$ is complex conjugation*

2. $\langle k \cdot x, y \rangle = \langle x, k^{-1} \cdot y \rangle$ $\forall x, y \in X$, $k \in K$

Definition 17 *Suppose V is a module for H (as defined on page 2.3). A Hermitian form $\langle \cdot, \cdot \rangle_H$ on V is* invariant *if*

$$\langle x \cdot u, v \rangle_H = \langle u, x^* \cdot v \rangle_H,\ \forall u, v \in V,\ x \in H$$

If an invariant form is positive definite, then the representation is said to be unitary.

The operation x^* here is the anti-involution

$$w^* = w^{-1}$$

$$f^* = -f + \sum_{\beta \in R^+} \mathbf{c}(\beta) f(\beta) s_\beta$$

inherited from the affine Hecke algebra.

The computation is facilitated by alternate presentation of H, called the Drinfeld presentation.

3.4.2 Drinfeld Presentation

Given an element f of $\mathfrak{a}$, define $\tilde{f} \in H$ by

$$\tilde{f} = f - \frac{1}{2} \sum_{\beta \in R^+} c(\beta) f(\beta) s_\beta$$

where R^+ is a fixed set of positive roots and

$$\mathbf{c}(\beta) = \begin{cases} 1 & \beta \text{ short} \\ k & \beta \text{ long} \end{cases}$$

Lemma 18 *Given $\tilde{f}, \tilde{g} \in H$, the following identities hold:*

1. $s_\alpha \tilde{f} = \widetilde{s_\alpha f} \quad \forall s_\alpha \in \mathbb{C}[W]$
2. $[\tilde{f}, \tilde{g}] = \left[\frac{1}{2} \sum_{\beta \in R^+} \mathbf{c}(\beta) f(\beta) s_\beta, \frac{1}{2} \sum_{\beta \in R^+} \mathbf{c}(\beta) g(\beta) s_\beta\right]$
3. $\tilde{f}^* = -\tilde{f}$

The proofs of these identities are one-line computations.

With the first two relations, the elements $\tilde{f}$ and $\mathbb{C}[W]$ form a presentation of H called the Drinfeld presentation [7]. It is critical to the proof of preservation of unitarity both because of identity (3) in lemma 18, and also because of the following important technical lemma:

Lemma 19 *The elements of H of the form $\tilde{\epsilon}_l$, where ϵ_l is a basis element of H, act by $\pi(\Omega^{\mathfrak{p}}_{0,l})$ on $\widetilde{F}_\delta(X)$ where*

$$\Omega^{\mathfrak{p}}_{0,l} = \sum_{E \in B \cap \mathfrak{p}} (E)_0 \otimes (E)_l$$

Proof of lemma: First rewrite the operator $\Omega^{\mathfrak{p}}_{0,l}$ as

$$\pi(\Omega^{\mathfrak{p}}_{0,l}) = \pi(\Omega_{0,l} - \Omega^{\mathfrak{k}}_{0,l})$$

Using the definition of the ϵ_i action:

$$= \pi(\epsilon_l) - n + \sum_{i<l} \pi(s_{i,l}) - \Omega^{\mathfrak{k}}_{0,l}$$

From the computation on page 3.3, the action of $-\Omega^{\mathfrak{k}}_{0,l}$ can be replaced thus:

$$= \pi(\epsilon_l) - n + \sum_{i<l} \pi(s_{i,l}) + \frac{k}{2}\pi(\bar{s}_l) + n + \sum_{1\le i\ne l\le n} \Omega^{\mathfrak{k}}_{i,l}$$
$$= \pi(\epsilon_l) + \sum_{i<l} \pi(s_{i,l}) + \frac{k}{2}\pi(\bar{s}_l) + \sum_{i\ne l} \Omega^{\mathfrak{k}}_{i,l}$$

It is easy to check by computation on basis vectors that $\pi(\Omega^{\mathfrak{k}}_{i,j}) = -\frac{1}{2}\pi(s_{i,j} + \bar{s}_j s_{i,j} \bar{s}_j)$, which yields:

$$= \pi(\epsilon_l) + \frac{k}{2}\pi(\bar{s}_l) + \frac{1}{2}\sum_{i<l} \pi(s_{i,l}) - \frac{1}{2}\sum_{i>l} \pi(s_{i,l}) - \frac{1}{2}\sum_{i\ne l} \pi(\bar{s}_j s_{i,j} \bar{s}_j)$$

But since $\bar{s}_i$ is a reflection in the long root, and the rest of the terms are reflections in short roots, this is exactly the formula for $\tilde{\epsilon}_i$.

□

3.4.3 Preservation of Unitarity

Lemma 20 *Suppose $\langle\cdot,\cdot\rangle_X$ is an invariant form on a $(\mathfrak{g}, K)$-module X.*

1. *This induces an H-invariant form $(\cdot,\cdot)$ on $\widetilde{F}_\delta(X)$.*
2. *If $\langle\cdot,\cdot\rangle_X$ is positive definite and nondegenerate, then $(\cdot,\cdot)$ is positive definite and nondegenerate.*

Proof of lemma: Begin by noticing that $\widetilde{F}_\delta(X) = \mathrm{Hom}_{K_\mathbb{R}}(\mu, X_{\mu\otimes\mu_0^*} \otimes (\mathbb{C}^{2n})^{\otimes n})$ is the trivial isotypic subspace of $\mathrm{Hom}_\mathbb{C}(\mu, X_{\mu\otimes\mu_0^*} \otimes (\mathbb{C}^{2n})^{\otimes n})$. This latter space is somewhat more manageable, since in this case the vectors such that $f(1)$ is of the form $x \otimes u_1 \otimes \ldots \otimes u_n$ span the space. The strategy will be to define a Hermitian form on these vectors and extend it by linearity, then restrict to $\widetilde{F}_\delta(X)$.

Since $\mathbb{C}^{2n}$ is finite dimensional, we can fix a form on it such that

$$(k\cdot x, y) = (x, k^{-1}\cdot y), \quad \forall x, y \in \mathbb{C}^{2n},\ k \in K_\mathbb{R}$$
$$(E\cdot x, y) = (x, E\cdot y), \quad \forall x, y \in \mathbb{C}^{2n},\ E \in \mathfrak{p}_\mathbb{R}$$

Suppose $f(1) = x \otimes u_1 \otimes \ldots \otimes u_n$ and $g(1) = y \otimes u_1 \otimes \ldots \otimes u_n$. Define $\langle f, g\rangle_1$ by

$$\langle f, g\rangle = \langle x, y\rangle_X (u_1, v_1)\ldots(u_n, v_n)$$

After extending to $\mathrm{Hom}_\mathbb{C}(\mu, X_{\mu\otimes\mu_0^*} \otimes (\mathbb{C}^{2n})^{\otimes n})$ by linearity, it is clear that this is a $K_\mathbb{R}$-invariant Hermitian form, since both $\langle\cdot,\cdot\rangle_X$ and $(\cdot,\cdot)$ are $K_\mathbb{R}$-invariant. Because it is $K_\mathbb{R}$-invariant, it induces a well-defined form on $\widetilde{F}_\delta$. It remains to show that this induced form is H-invariant and preserves nondegeneracy.

1. It suffices to show that the form is invariant on basis elements of $\mathrm{Hom}_{\mathbb{C}}(\mu, X_{\mu\otimes\mu_0^*} \otimes (\mathbb{C}^{2n})^{\otimes n})$, for each generator of H. Given $f, g \in \mathrm{Hom}_{\mathbb{C}}(\mu, X_{\mu\otimes\mu_0^*} \otimes (\mathbb{C}^{2n})^{\otimes n})$ as above, and proceeding generator by generator:

   $s_{i,j}$: Since $\pi(s_{i,j})$ simply interchanges the i and j coordinates, and then negates the result, the computation is straightforward:

$$\begin{aligned}\langle \pi(s_{i,j})f, g\rangle &= \langle -\Omega_{i,j} f, g\rangle \\ &= -\langle x, y\rangle (u_1, v_1)\ldots(u_j, v_i)\ldots(u_i, v_j)\ldots(u_n, v_n) \\ &= \langle f, -\Omega_{i,j} g\rangle \\ &= \langle f, \pi(s_{i,j})g\rangle\end{aligned}$$

   $\bar{s}_i$: From the definition of ξ we know that $\xi = \xi^{-1}$; it is also clear that $\xi \in K$, since it is fixed by θ. Thus

$$(\xi x, y) = (x, \xi y)$$

   Since $\pi(\bar{s}_i)$ is multiplication by $-\xi$ in the i^{th} coordinate, it is clear that $\langle \pi(\bar{s}_i)f, g\rangle = \langle f, \pi(\bar{s}_i)g\rangle$.

   $\tilde{\epsilon}_i$: Since $\tilde{\epsilon}_i$ acts by $\pi(\Omega^{\mathfrak{p}}_{i,j})$, we have that

$$\begin{aligned}\langle \tilde{\epsilon}_i \cdot f, g\rangle &= \langle \Omega^{\mathfrak{p}}_{0,i}(x \otimes u_1 \otimes \ldots \otimes u_n), (y \otimes v_1 \otimes \ldots \otimes v_n)\rangle \\ &= \sum_{E\in B} \langle Ex, y\rangle_X (u_1, v_1)\ldots(E^* u_i, v_i)\ldots(u_n, v_n) \\ &= -\sum_{E\in B} \langle x, Ey\rangle_X (u_1, v_1)\ldots(u_i, E^* v_i)\ldots(u_n, v_n) \\ &= -\langle (x \otimes u_1 \otimes \ldots \otimes u_n), \Omega^{\mathfrak{p}}_{0,i}(y \otimes v_1 \otimes \ldots \otimes v_n)\rangle \\ &= -\langle f, \tilde{\epsilon}_i \cdot g\rangle\end{aligned}$$

2. Suppose $\langle \cdot, \cdot\rangle_X$ is positive definite and nondegenerate. If $\langle \cdot, \cdot\rangle$ is degenerate on $\widetilde{F}_\delta$, then there is some $\phi \in \widetilde{F}_\delta$ such that $\langle \phi, \phi\rangle = 0$. Fix a basis $\{e_i\}$ of $\mathbb{C}^{2n}$ orthonormal with respect to $(\cdot, \cdot)$. Then $\phi(1) = \sum_i x_i \otimes e_{i_1} \otimes \ldots \otimes e_{i_n}$, where $x_i \in X$, and the sum is over all possible tensor products of length n of basis vectors. Then we have

$$\begin{aligned}\langle \phi, \phi\rangle &= \sum_{i,j} \langle x_i, x_j\rangle_X (e_{i_1}, e_{j_1})\ldots(e_{i_n}, e_{j_n}) \\ &= \sum_{i,j} \langle x_i, x_j\rangle_X \delta_{i,j} \\ &= \sum_i \langle x_i, x_i\rangle_X\end{aligned}$$

   But since $\langle \cdot, \cdot\rangle_X$ is nondegenerate and positive definite, this is nonzero.

□

3.5 Preservation of Irreducibility

Preservation of irreducibility is automatic, once preservation of unitarity is known, due to the following result paraphrased from [10].

Lemma 21 *Let $X^{\mathbb{R}}_{\delta}(\nu)$ denote the principal series of the form $\mathrm{Ind}_{M_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}}(\delta \otimes e^{\nu} \otimes 1)$ and $\bar{X}^{\mathbb{R}}_{\delta}(\nu)$ denote its unique irreducible quotient. Then there exists a unique invariant Hermitian form $\langle\cdot,\cdot\rangle_X$ on $X^{\mathbb{R}}_1(\nu)$ such that the following sequence is exact:*

$$0 \longrightarrow \mathrm{rad}\langle\cdot,\cdot\rangle_X \longrightarrow X^{\mathbb{R}}_1(\nu) \longrightarrow \bar{X}^{\mathbb{R}}_1(\nu) \longrightarrow 0$$

Since $\widetilde{F}_{\delta}$ is an exact functor, that implies

$$0 \longrightarrow \widetilde{F}_{\delta}(\mathrm{rad}\langle\cdot,\cdot\rangle_X) \longrightarrow X_1(\nu) \longrightarrow \widetilde{F}_{\delta}(\bar{X}^{\mathbb{R}}_1(\nu)) \longrightarrow 0$$

Or, because $\widetilde{F}_{\delta}$ maps standard modules to standard modules and induces a form $\langle\cdot,\cdot\rangle$ on $X_1(\nu)$, we have

$$0 \longrightarrow \mathrm{rad}\langle\cdot,\cdot\rangle \longrightarrow X_1(\nu) \longrightarrow \widetilde{F}_{\delta}(\bar{X}^{\mathbb{R}}_1(\nu)) \longrightarrow 0$$

Since $\bar{X}^{\mathbb{R}}_1(\nu)$ is non-zero, there is some vector $x \in X_1$ such that $\langle x, x\rangle \neq 0$; a vector $f \in X_1(\nu)$ such that $f(1) = x \otimes e_1 \otimes \ldots \otimes e_n$ will also have the property that $\langle f, f\rangle \neq 0$, so $\widetilde{F}_{\delta}(\bar{X}^{\mathbb{R}}_1(\nu)) \neq 0$.

This completes the proof of of the main result.

CHAPTER 4

THE TYPE A, NON-SPHERICAL CASE

4.1 Introduction

There are two obvious directions in which to extend the results of Ciubotaru and Trapa – either to include additional groups, as in the case of $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$, or to include nonspherical representations of linear groups. This section of the thesis will focus on a first attempt at the latter.

Suppose $F = \mathbb{Q}_p$, $G_F = \mathrm{GL}(n, F)$ and $K = \mathrm{GL}(n, \mathbb{Z}_p)$, the maximal compact subgroup of G_F. Say A is the maximal split torus of G_F, and $A^0 = A \cap K$. Fix a nontrivial character $\chi^0 : A^0 \to \mathbb{C}^\times$.

It is possible to define minimal principal series representations of G_F in the same way as for real groups; see definition 3. Given a character χ of A such that $\chi|_{A^0} = \chi^0$, extended to a representation of a minimal parabolic P, consider the representation $\mathrm{Ind}_P^{G_F}(\chi)$. As in the real case, these representations have interesting irreducible subquotients.

The category we are interested in studying is the category $\mathcal{C}(\chi^0)$, the full subcategory of G_F representations whose irreducible objects are the irreducible subquotients of the $\mathrm{Ind}_P^{G_F}(\chi)$. This category can actually be studied using Hecke algebra techniques similar to those available in the spherical case.

A slightly modified Hecke algebra is necessary. Given a compact open subgroup $J \subset G_F$ and a J-type ρ, $\mathcal{H}(G_F//J, \rho)$ is the convolution algebra of functions $f : G_F \to \mathbb{C}$ such that

$$f(k_1, g, k_2) = \rho(k_1)\rho(g)\rho(k_2)\ \forall k_1, k_2 \in J\ \forall g \in G_F$$

Roche proved in [15] that it is possible to choose a particular compact open subgroup $J \subset G_F$ and a J-type ρ such that $\rho|_{A^0} = X^0$ so that there is a categorical equivalence between $\mathcal{C}(\chi^0)$ and the category of $\mathcal{H}(G_F//J, \rho)$-modules. A further reduction is possible, due to Howe and Moy [8]. The algebra $\mathcal{H}(G_F//J, \rho)$ is isomorphic to an Iwahori-Hecke algebra of an endoscopic group of G_F.

The particular case of interest in this chapter is when κ is a quadratic character of $\mathbb{Z}_p^\times$, and $\chi^0 : A^0 = (\mathbb{Z}_p^\times)^n \to \mathbb{C}$ is of the form:

$$\chi^0 = 1 \otimes \cdots \otimes 1 \otimes \kappa \otimes \cdots \otimes \kappa$$

where are p factors of A^0 act by the trivial representation and q copies act by κ. In this case, the algebra is reduced to the Iwahori-Hecke algebra $\mathcal{H}(\mathrm{GL}(n-q, F) \times \mathrm{GL}(q, F))$, for some $q <= n$. This suggests that we would like to produce a functor whose target category is $\mathbb{H}_{n-q} \times \mathbb{H}_q$ modules.

Suppose $G_\mathbb{R} = \mathrm{GL}(n, \mathbb{R})$, $K_\mathbb{R} = SO(n)$, its maximal compact subgroup, and $P_\mathbb{R} = M_\mathbb{R} A_\mathbb{R} N_\mathbb{R}$ is the Borel subgroup of upper triangular matrices.

Since $M_\mathbb{R} \cong \big(\mathbb{Z}/2\mathbb{Z}\big)$, a character of $M_\mathbb{R}$ will be a tensor product of n copies of sgn or the trivial representation. We would like to choose ω to be a character analogous to the choice of χ^0 above, so select the character

$$\chi^0 = 1 \otimes \cdots \otimes 1 \otimes \mathrm{sgn} \otimes \cdots \otimes \mathrm{sgn}$$

which again has $n-q$ copies of the trivial representation and q copies of sgn. For this choice of ω, a principal series representation $I_\omega(\nu)$ will have minimal $K_\mathbb{R}$-type $\wedge^p \mathbb{C}^n$ (see [9]).

4.2 Obstruction

The major obstruction in extending Ciubotaru and Trapa's results to other nonlinear groups, as well as to nonspherical principal series representations of linear groups, is encapsulated in the nonspherical type A case.

Suppose $G_\mathbb{R} = \mathrm{GL}(n, \mathbb{R})$, $K_\mathbb{R} = \mathrm{SO}(n)$, its maximal compact subgroup, and $P_\mathbb{R} = M_\mathbb{R} A_\mathbb{R} N_\mathbb{R}$ is the Borel subgroup of upper triangular matrices. Consider $I_\omega(\nu)$, a principal series representation of $\mathrm{GL}(n, \mathbb{R})$ with nontrivial minimal $K_\mathbb{R}$-type μ.

Naïvely, one might try to construct an action of some graded affine Hecke algebra on the following space:

$$\mathrm{Hom}_{K_\mathbb{R}}\Big(\mu, I_\omega(\nu) \otimes \big(\mathbb{C}^n\big)^{\otimes n}\Big) \tag{4.1}$$

In this case, the possible minimal $K_\mathbb{R}$-types for the principal series $I_\omega(\nu)$ will be of the form $\wedge^p \mathbb{C}^n$, and the Hecke algebra action should be by the algebra $\mathbb{H}_p \times \mathbb{H}_q$, where $p+q = n$. Ideally, the space in (4.1) should be a standard module for $\mathbb{H}_p \times \mathbb{H}_q$, i.e., it should be of the form:

$$\mathbb{I}_1(\nu) = (\mathbb{H}_p \times \mathbb{H}_q) \otimes_{\mathrm{S}(\mathfrak{a})} \mathbb{C}_\nu$$

where $\mathbb{C}_\nu$ is a one-dimensional representation of $\mathrm{S}(\mathfrak{a})$ with infinitesimal character ν. This space is isomorphic, as a vector space, to $\mathbb{C}[\mathrm{S}_p \times \mathrm{S}_q]$, and so has dimension $p!q!$.

The problem with (4.1) is most vividly illustrated in the case $n = 2$, $p = 1$. In that case, $K_{\mathbb{R}} = \mathrm{O}(n)$, $P_{\mathbb{R}}$ is the upper triangular subgroup, and

$$M_{\mathbb{R}} = K_{\mathbb{R}} \cap P_{\mathbb{R}} = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

There are four irreducible representations of $M_{\mathbb{R}}$; denote these by ordered pairs $(1,1)$, $(1, \mathrm{sgn})$, $(\mathrm{sgn}, 1)$, and $(\mathrm{sgn}, \mathrm{sgn})$, where sgn is the sign representation of $\mathbb{Z}/2\mathbb{Z}$.

The space in (4.1) now becomes:

$$\begin{aligned} \mathrm{Hom}_{K_{\mathbb{R}}}\left(\wedge^1 \mathbb{C}^2, I_\omega(\nu) \otimes \left(\mathbb{C}^2\right)^{\otimes 2}\right) &= \mathrm{Hom}_{K_{\mathbb{R}}}\left(\mathbb{C}^2, I_\omega(\nu) \otimes \left(\mathbb{C}^2\right)^{\otimes 2}\right) \\ &\cong \mathrm{Hom}_{K_{\mathbb{R}}}\left(1, I_\omega(\nu) \otimes \left(\mathbb{C}^2\right)^{\otimes 2} \otimes \left(\mathbb{C}^2\right)^*\right) \\ &\cong \mathrm{Hom}_{K_{\mathbb{R}}}\left(1, I_\omega\left(\nu \otimes \left(\mathbb{C}^2\right)^{\otimes 2} \otimes \left(\mathbb{C}^2\right)^*\right)\right) \\ &\cong \mathrm{Hom}_{M_{\mathbb{R}}}\left((1,1), (\mathrm{sgn}, 1) \otimes \left(\mathbb{C}^2\right)^{\otimes 2} \otimes \left(\mathbb{C}^2\right)^*\right) \end{aligned} \tag{4.2}$$

Explicitly decomposing the space on the right-hand side as an $M_{\mathbb{R}}$ representation yields:

$$\begin{aligned} &(\mathrm{sgn}, 1) \otimes \left(\mathbb{C}^2\right)^{\otimes 2} \otimes \left(\mathbb{C}^2\right)^* \\ &\quad = (\mathrm{sgn}, 1) \otimes \big[(1,1) \oplus (1,1) \oplus (\mathrm{sgn}, \mathrm{sgn}) \oplus (\mathrm{sgn}, \mathrm{sgn})\big] \otimes \big[(1, \mathrm{sgn}) \oplus (\mathrm{sgn}, 1)\big] \\ &\quad = (\mathrm{sgn}, \mathrm{sgn}) \oplus (\mathrm{sgn}, \mathrm{sgn}) \oplus (\mathrm{sgn}, \mathrm{sgn}) \oplus (\mathrm{sgn}, \mathrm{sgn}) \\ &\qquad\quad (1,1) \oplus (1,1) \oplus (1,1) \oplus (1,1) \end{aligned}$$

The space in (4.2) is thus four-dimensional, rather than the one-dimensional, as desired. For general n and p, the dimension of this space is difficult to compute, but it is always finite.

The question, then, is how to reduce the space to the appropriate size. The approach used in this paper is to construct an action of $\mathbb{H}_p \times \mathbb{H}_q$ on the space, then project to the portion of it with the correct infinitesimal character. It can be show that, except at particular "bad" infinitesimal characters, composition with this projection will produce a functor, which carries standard modules of $G_{\mathbb{R}}$ to standard modules of $\mathbb{H}_p \times \mathbb{H}_q$.

4.3 Construction of the $\mathbb{H}_p \times \mathbb{H}_q$-Action

The construction of the $\mathbb{H}_p \times \mathbb{H}_q$-action is similar to, but simpler than, the construction of the $\mathbb{H}_{\mathrm{B}_n}(\mathbf{c})$ action in the metaplectic case. Fix a bilinear form on $\mathfrak{g}$, given by:

$$\langle X, Y \rangle = \mathrm{tr}(XY)$$

Also fix a basis B of $\mathfrak{g}$, together with a duality operation $E \mapsto E^*$, so that B is self-dual in the sense that:

$$\langle X, Y \rangle = \begin{cases} 1 & \text{if } Y = X^* \\ 0 & \text{otherwise} \end{cases}$$

for all $X, Y \in B$. Then define the operator $\Omega_{i,j}$ on $X \otimes \left(\mathbb{C}^n\right)^{\otimes n}$ to be:

$$\Omega_{i,j} = \sum_{E \in B} (E)_i \otimes (E^*)_j$$

where $(E)_i$ indicates multiplication by E in the i^{th} factor of the tensor product $X \otimes \left(\mathbb{C}^n\right)^{\otimes n}$, with X counting as the 0^{th} factor.

The graded affine Hecke algebra $\mathbb{H}_p \times \mathbb{H}_q$ is isomorphic the graded affine Hecke algebra $\mathbb{H}_{A_{p+1} \times A_{q+1}}(c \equiv 1)$, so it is isomorphic (as a vector space) to $\mathbb{C}[S_p \times S_q] \otimes S(\mathfrak{a})$, together with the relations:

$$\begin{aligned} s_{i,i+1}\epsilon_i - \epsilon_{i+1}s_{i,i+1} &= 1 & \\ s_{i,i+1}\epsilon_j - \epsilon_j s_{i,i+1} &= 0 & \text{if } j \neq i, i+1 \end{aligned}$$

where $S_p \times S_q$ is regarded as a subgroup of S_{p+q}. In this case $s_{i,i+1}$ is the transposition of the i and $i+1$ positions. The ϵ_i are given by $\epsilon_i = (0, \ldots, 1, \ldots, 0)$, i.e. the vector which is zero in all but the i^{th} coordinate; the coordinates here are such that $\epsilon_i - \epsilon_{i+1}$ is a simple co-root.

What makes the action of $\mathbb{H}_p \times \mathbb{H}_q$ slightly simpler to deal with than the action in the metaplectic case is the following lemma:

Lemma 22 *The action of $\Omega_{i,j}$, $i, j > 0$ on $X \otimes \left(\mathbb{C}^n\right)^{\otimes n}$ is reflection in the i, j coordinates.*

Proof of lemma: As in the previous chapter, it is useful to reduce to the case of $\Omega_{1,2}$ acting on $\mathbb{C}^n \otimes \mathbb{C}^n$. A good choice of basis B is the set of matrices $E_{i,j}$; these are matrices with 1 in the i^{th} row and j^{th} column and 0 elsewhere. With this choice of basis, the operator $\Omega_{1,2}$ becomes:

$$\sum_{1 \leq i,j \leq n} E_{i,j} \otimes E_{j,i}$$

Implicitly, this also fixes a basis $\{e_i\}_{i=1}^n$ of $\mathbb{C}^n$, and thus a basis $\{e_i \otimes e_j\}_{i.j=1}^n$ of $\mathbb{C}^n \otimes \mathbb{C}^n$. But the lemma is then clear from explicit computation:

$$\left(\sum_{1 \leq i,j \leq n} E_{i,j} \otimes E_{j,i}\right) e_k \otimes e_l = (E_{l,k} \otimes E_{k,l})(e_k \otimes e_l) = e_l \otimes e_k$$

□

In other words, the type A_n case is special in that there is a well defined action of $\mathbb{H}_p \times \mathbb{H}_q$ on $X \otimes \left(\mathbb{C}^n\right)^{\otimes n}$. More precisely, the following lemma holds:

Lemma 23 *For any object X in $\mathcal{HC}_\omega(\nu)$, there is a well-defined action π of $\mathbb{H}_p \times \mathbb{H}_q$ on $X \otimes \left(\mathbb{C}^n\right)^{\otimes n}$ given by:*

$$\pi(s_{i,i+1}) = -\Omega_{i,j}$$
$$\pi(\epsilon_l) = \sum_{i=0}^{l} \Omega_{i,l} + \frac{n-1}{2}$$

Proof of lemma: Since $\mathbb{H}_p \times \mathbb{H}_q$ embeds in $\mathbb{H}_n$, it suffices to prove the lemma for $p = n$, $q = 0$. These computations are identical, line by line, with the corresponding computations in the proof of lemma 12.

□

Unlike the metaplectic case, the central shift here by $\frac{n-1}{2}$ is not required to make the action well-defined. However, it is required to make the functor preserve infinitesimal character, which will become critical later.

This lemma makes it possible to define a functor $\mathcal{F}_{\mu,\lambda}$ from $\mathcal{HC}_\mu(\lambda)$ to $\mathbb{H}_p \times \mathbb{H}_q$-mod.

Theorem 24 *There is an exact functor $\mathcal{F}_{\mu,\lambda}$ from $\mathcal{HC}_\mu(\lambda)$ to $\mathbb{H}_p \times \mathbb{H}_q$-mod given by*

$$\mathcal{F}_\mu(X) = \mathrm{Hom}_{K_\mathbb{R}}\left(\mu, X \otimes \left(\mathbb{C}^n\right)^{\otimes n}\right) \tag{4.3}$$

Given an infinitesimal character λ of $A_\mathbb{R}$, there is an exact functor $\mathcal{HC}_\mu(\lambda)$ to $\mathbb{H}_p \times \mathbb{H}_q$-mod given by

$$\mathcal{F}_{\mu,\lambda}(X) = \mathcal{F}_\mu(X)|_\lambda \tag{4.4}$$

where the restriction indicates projection onto the portion of $\mathcal{F}_\mu(X)$ with $\mathbb{H}_p \times \mathbb{H}_q$ central character λ; this functor carries standard modules to standard modules for almost all choices of central character λ.

Exactness follows from the fact that the $\Omega_{i,j}$ commutes with the action of $K_\mathbb{R}$, which is proved in exactly the same way as in the metaplectic case. It remains to prove that this functor carries standard modules to standard modules.

4.4 Eigenvalues of the $S(\mathfrak{a})$ Action

4.4.1 Basis of $\mathcal{F}_{\mu,\lambda}(I_\mu(\lambda))$

It will be convenient to fix some notation. For the orthnormal basis B of $\mathfrak{g}$, choose the basis

$$B = \{E_{i,j}\}_{1 \le i,j \le n} \tag{4.5}$$

where $E_{i,j}$ is the matrix with 1 in the i^{th} row and j^{th} column and 0 elsewhere. Also fix the $\mathbb{C}^n$ basis $\{e_i\}_{i=1}^n$ to be the basis making $P_\mathbb{R}$ upper triangular.

By Frobenius reciprocity and a Mackey isomorphism,

$$\mathrm{Hom}_{K_{\mathbb{R}}}\left(\wedge^p \mathbb{C}^n, X \otimes \left(\mathbb{C}^n\right)^{\otimes n}\right) \cong \mathrm{Hom}_{M_{\mathbb{R}}}\left(1, \left(\mathrm{sgn}^{\otimes p} \otimes 1^{\otimes q}\right) \otimes \left(\mathbb{C}^n\right)^{\otimes n} \otimes \left(\wedge^p \mathbb{C}^n\right)^*\right) \quad (4.6)$$

Though the space on the left is quite complicated, the space on the right has a straightforward basis. Given an element of $\left(\mathbb{C}^n\right)^{\otimes n}$ of the form $e_{i_1} \otimes \cdots \otimes e_{i_n}$, there is at most one possible choice for an element w of $\left(\wedge^p \mathbb{C}^n\right)^*$ such that $\phi'(1) = v \otimes e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes w$ is in the space. A basis of the space can thus be parameterized by $\left\{e_{i_1} \otimes \cdots \otimes e_{i_n}\right\}$. Denote these basis elements by $\{\phi'_{i_1 \dots i_n} = \phi'_{\bar{i}}\}$. Since Frobenius reciprocity is a canonical isomorphism, denote by $\phi_{\bar{i}}$ the element of the space on the left-hand side corresponding to $\phi'_{\bar{i}}$.

There is a natural ordering on these basis vectors, given by the lexicographic ordering. Suppose $\phi_{\bar{a}} \neq \phi_{\bar{b}}$, and i is the smallest integer such that $a_i \neq b_i$. Then $\phi_{\bar{a}} \leq \phi_{\bar{b}}$ if and only if $a_i \leq b_i$.

4.4.2 Action of $\Omega_{0,l}$

The action of $\Omega_{i,j}$, with $i > 0$, is relatively straightforward in this case, but the action of $\Omega_{0,l}$ requires some careful investigation.

For the sake of simplicity, replace

$$\mathrm{Hom}_K\Big(\mu, \mathrm{Ind}_{P_{\mathbb{R}}}^{G_{\mathbb{R}}}\left(\Big(\nu \otimes e^{\lambda} \otimes \mathbb{C}\Big) \otimes \det \otimes (\mathbb{C}^n)^{\otimes n}\right)\Big) \tag{4.7}$$

with the isomorphic space

$$\mathrm{Hom}_K\Big(\mu, \mathrm{Ind}_{P \times G \times \cdots \times G}^{G \times G \times \cdots \times G}\left(\Big(\nu \otimes e^{\lambda} \otimes \mathbb{C}\Big) \otimes \det \otimes (\mathbb{C}^n)^{\otimes n}\right)\Big) \tag{4.8}$$

where K and $\mathfrak{g}$ act diagonally on the induced representation. Denote the action on the i^{th} coordinate by π_i, where the indexing begins at 0. That space is isomorphic under Frobenius reciprocity to the space

$$\mathrm{Hom}_M\left(\mu, (\nu \otimes \det) \otimes (\mathbb{C}^n)^{\otimes n}\right) \tag{4.9}$$

Compute the action of $\Omega_{0,l}$ by proceeding element by element in B. I'll skip the proofs for most of them, since only one differs from the existing $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ case.

For the remainder of this section, $\psi'_{a_1 \dots a_n}$ will denote the basis element of (4.9) corresponding to the basis element $\phi_{a_1 \dots a_n}$ of (4.7) under the chain of isomorphisms above.

Following this chain of isomorphisms, the basis $\{\phi_{\bar{\imath}}\}$ corresponds to a basis $\{\psi_{\bar{\imath}}\}$ of (4.8) and $\{\psi'_{\bar{\imath}}\}$ of (4.9). Explicitly, suppose $\psi'_{\bar{\imath}}$ is the element of the space (4.9) such that

$$\psi'_{\bar{\imath}}(1) = w \otimes e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes v_{\bar{\imath}} \tag{4.10}$$

where $v_{\bar{\imath}}$ is an element of the unique one-dimensional subspace of $\det \otimes \left(\wedge^p \mathbb{C}^n\right)^*$ that makes $\psi'_{\bar{\imath}}$ a homomorphism. The corresponding element $\psi_{\bar{\imath}}$ of (4.8) is the homomorphism such that

$$\psi_{\bar{\imath}}(1)(x_0, \ldots, x_{n+1}) = \pi_1(x_1^{-1}x_0) \ldots \pi_{n+1}(x_{n+1}^{-1}x_0)\psi_{\bar{\imath}}^{\Delta}(1)(x_0) \tag{4.11}$$

where $\psi_{\bar{\imath}}^{\Delta}$ is defined, using the Iwasawa decomposition $G_{\mathbb{R}} = K_{\mathbb{R}}M_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}$, to be:

$$\psi_{\bar{\imath}}^{\Delta}(1)(kman) = (man)^{-1}\psi'_{\bar{\imath}}(1) \tag{4.12}$$

In fact, $\psi_{\bar{\imath}}^{\Delta}(1)$ is the element of the induced representation in (4.8) corresponding to $\psi'_{\bar{\imath}}(1)$ under Frobenius reciprocity, so it is $K_{\mathbb{R}}$ invariant and well-defined. Also note that the entire construction depends only on $\bar{\imath}$.

The computation of the action of $\Omega_{0,l}$ is accomplished by breaking it up into the actions of the $\mathfrak{n}$, $\mathfrak{a}$, and $\bar{\mathfrak{n}}$ terms.

Lemma 25 *Suppose $E = E_{ii} \in B \cap \mathfrak{h}$. Then*

$$((E)_0 \otimes (E)_l)\psi'_{a_1\ldots a_n}(f)(1) = \begin{cases} (\lambda+\rho)(E_{ii})\psi'_{a_1\ldots a_n}(f)(1) & \text{if } a_l = i \\ 0 & \text{otherwise} \end{cases}$$

Lemma 26 *Suppose $E = E_{ij} \in B \cap \mathfrak{n}$. Then*

$$((E)_0 \otimes (E)_l)\psi'_{a_1\ldots a_n} = 0$$

Lemma 27 *Suppose $E = E_{ij} \in B \cap \bar{\mathfrak{n}}$. Then*

$$((E)_0 \otimes (E)_l)\psi'_{\vec{a}}(f)(1) = \left(-\pi_l(E_{jj}) + \pi_l(E_{ji}) \sum_{m \neq l} (\pi_m(E_{ji} - E_{ij})) \right) \psi'_{\vec{a}}(f)(1)$$

Proof of lemma: Since $E_{ij} \in \bar{\mathfrak{n}}$, $-\theta E_{ij} = E_{ji} \in \mathfrak{n}$. Set $H = E_{ij} + \theta(E_{ij}) = E_{ij} - E_{ji} \in \mathfrak{k}$. Since $(E_{ji})_0 \otimes (E_{ij})_l$ acts by 0 on $\psi'_{\vec{a}}$, the operators $((E)_0 \otimes (E)_l)$ and $((H)_0 \otimes (E)_l)$ are equivalent.

Then:

$$((H)_0 \otimes (E)_l)\psi'_{\vec{a}}(f)(1) = \left.\frac{d^2}{duds}\right|_{u=s=0} \pi_l(e^{sE}) \prod_{m=1}^{n} \pi_m\left(e^{-uH}\right) \psi'_{\vec{a}}(f)\left(e^{-uH}\right)$$

The following step differs from [6], and great care is needed.

The vector $\psi'_{\vec{a}}$ is an element of the space

$$\mathrm{Hom}_{K_\mathbb{R}}\Big(\mu, \mathrm{Ind}_P^G((\eta \otimes e^\lambda \otimes 1) \otimes \det \otimes (\mathbb{C}^n)^{\otimes n})\Big)$$

Suppose π_μ is the representation on μ, and π_X is the representation on the induced module. Then since $e^{uH} \in K_\mathbb{R}$ and $\psi'_{\vec{a}}$ is a $K_\mathbb{R}$-homomorphism:

$$\psi'_{\vec{a}}(f)\left(e^{-uH}\right) = \pi_X(e^{uH})\psi'_{\vec{a}}(f)(1)$$

Continuing the computation:

$$\begin{aligned}
&((H)_0 \otimes (E)_l)\phi_{\vec{a}}(f)(1) \\
&\quad = \frac{d^2}{duds}\bigg|_{u=s=0} \pi_l\left(e^{sE}\right) \prod_{m=1}^{n} \pi_m\left(e^{-uH}\right) \pi_X\left(e^{uH}\right) \psi'_{\vec{a}}(f)(1) \\
&\quad = \frac{d}{du}\pi_l(E) \prod_{m=1}^{n} \pi_m\left(e^{-uH}\right) \pi_X\left(e^{uH}\right) \psi'_{\vec{a}}(f)(1) \\
&\quad = \pi_l(E)\left(\pi_X(H) + \sum_{m=1}^{n} \pi_m(-H)\right)\psi'_{\vec{a}}(f)(1) \\
&\quad = \pi_l(E)\left(\sum_{m=1}^{n} \pi_m(-H)\right)\psi'_{\vec{a}}(f)(1) \\
&\qquad + \pi_X(H)\psi'_{\vec{a}}(f)(1) \\
&\quad = \pi_l(E)\left(\sum_{m=1}^{n} \pi_m(-H)\right)\psi'_{\vec{a}}(f)(1) \\
&\qquad + \psi'_{\vec{a}}(\pi_\mu(H)f)(1)
\end{aligned}$$

Since $\psi_{\vec{a}}$ is only nonzero in a single one-dimensional subspace of μ, $\psi'_{\vec{a}}(\pi_\mu(H)f)(1) = \psi_{\vec{a}}(\pi_\mu(H)f) = 0$. The final formula, then, is:

$$((H)_0 \otimes (E)_l)\phi_{\vec{a}}(f)(1) = \pi_l(E)\left(\sum_{m=1}^{n} \pi_m(-H)\right)\psi'_{\vec{a}}(f)(1)$$

In the present case, the only elements of $B \cap \bar{\mathfrak{n}}$ are of the form E_{ij} with $i > j$. Then the above formula becomes

$$\begin{aligned}
&((H_{ij})_0 \otimes (E_{ji})_l)\phi_{\vec{a}}(f)(1) \\
&\quad = \pi_l(E_{ji})\left(\sum_{m=1}^{n} \pi_m(-H_{ij})\right)\psi'_{\vec{a}}(f)(1) \\
&\quad = \left[\pi_l(E_{ji})\pi_l(E_{ji} - E_{ij}) + \pi_l(E_{ji})\left(\sum_{m\neq l}^{n} \pi_m(-H_{ij})\right)\right]\psi'_{\vec{a}}(f)(1) \\
&\quad = \left[-\pi_l(E_{jj}) + \pi_l(E_{ji}E_{ji}) + \pi_l(E_{ji})\left(\sum_{m\neq l}^{n} \pi_m(-H_{ij})\right)\right]\psi'_{\vec{a}}(f)(1)
\end{aligned}$$

$$= \left[-\pi_l(E_{jj}) + \pi_l(E_{ji}) \left(\sum_{m \neq l}^{n} \pi_m(-H_{ij}) \right) \right] \psi'_{\vec{a}}(f)(1)$$

□

4.5 Total ϵ_i Action

The total action is

$$\begin{aligned}
\epsilon_l \psi'_{\vec{a}}(f)(1) = &\sum_{i>j} \left(-\pi_l(E_{jj}) + \pi_l(E_{ji}) \sum_{m \neq l} (\pi_m(E_{ji} - E_{ij})) \right) \psi'_{\vec{a}}(f)(1) \\
&+ \left(\sum_{i=1}^{n} \pi_0(E_{ii}) \pi_l(E_{ii}) \right) \psi'_{\vec{a}}(f)(1) \\
&+ \left(\sum_{m<l} \sum_{E \in B} \pi_m(E) \pi_l(E) \right) \psi'_{\vec{a}}(f)(1) \\
&+ \frac{n-1}{2} \psi'_{\vec{a}}(f)(1)
\end{aligned}$$

The first two lines are the action of $\Omega_{0,l}$, while the third line is the action of $\sum_{0<i<l} \Omega_{i,l}$. The last line is the central shift. Rearranging terms a bit yields

$$\begin{aligned}
\epsilon_l \psi'_{\vec{a}}(f)(1) = &\sum_{i>j} \left(-\pi_l(E_{jj}) + \pi_l(E_{ji}) \sum_{m \neq l} (\pi_m(E_{ji} - E_{ij})) \right) \psi'_{\vec{a}}(f)(1) \\
&+ \left(\sum_{i=1}^{n} \pi_0(E_{ii}) \pi_l(E_{ii}) \right) \psi'_{\vec{a}}(f)(1) \\
&+ \left(\sum_{m<l} \sum_{i<j} \pi_m(E_{ij}) \pi_l(E_{ji}) \right) \psi'_{\vec{a}}(f)(1) \\
&+ \left(\sum_{m<l} \sum_{i=1}^{n} \pi_m(E_{ii}) \pi_l(E_{ii}) \right) \psi'_{\vec{a}}(f)(1) \\
&+ \left(\sum_{m<l} \sum_{i>j} \pi_m(E_{ij}) \pi_l(E_{ji}) \right) \psi'_{\vec{a}}(f)(1) \\
&+ \frac{n-1}{2} \psi'_{\vec{a}}(f)(1)
\end{aligned}$$

The terms are rearranged further below, grouped by the role they play in the action.

$$
\begin{aligned}
\epsilon_l \psi'_{\vec{a}}(f)(1) = &\left(\sum_{i=1}^{n} \pi_0(E_{ii})\pi_l(E_{ii})\right)\psi'_{\vec{a}}(f)(1) \\
&+ \left(\frac{n-1}{2} - \sum_{i>j} \pi_l(E_{jj})\right)\psi'_{\vec{a}}(f)(1) \qquad \text{(4.13a)} \\
&+ \left(\sum_{m<l}\sum_{i=1}^{n} \pi_m(E_{ii})\pi_l(E_{ii})\right)\psi'_{\vec{a}}(f)(1) \\
&+ \left(\sum_{m<l}\sum_{i<j} \pi_m(E_{ij})\pi_l(E_{ji})\right)\psi'_{\vec{a}}(f)(1) \\
&+ \left(\sum_{m>l}\sum_{i<j} \pi_l(E_{ij})\pi_m(E_{ji})\right)\psi'_{\vec{a}}(f)(1) \qquad \text{(4.13b)} \\
&+ \left(\sum_{m\neq l}\sum_{i<j} \pi_m(E_{ij})\pi_l(E_{ij})\right)\psi'_{\vec{a}}(f)(1) \qquad \text{(4.13c)}
\end{aligned}
$$

Lines 4.13a is the diagonal component of the action. The first line acts by $(\lambda+\rho)(E_{a_l a_l})$, since we are using normalized induction. The second line is somewhat more subtle. The second term acts by $-(n-a_l)$; this turns out to be $-\rho(E_{a_l a_l}) - \frac{n-1}{2}$. Total, then, the first two lines act by $\lambda(E_{a_l a_l})$. The third line will act only on a certain subspace of $\mathcal{F}_{\mu,\lambda}(X(\nu,\lambda))$. The inner sum acts by 1 if $a_m = a_l$; total, then, the third line acts by 1 for each $i < l$ such that $a_i = a_l$.

The action of lines 4.13b will only be non-zero if $\vec{a}$ has two indistinct terms. The terms in both lines act by E_{ij} in one factor of $\psi'_{\vec{a}}$ and E_{ji} on another. Importantly, though, the factor on which E_{ij} acts is to the left of the factor on which E_{ji} acts; since $i < j$, it will always lower the index of the basis vector in the first factor, and raise the index of the basis vector in the second. In other words, it will always send $\psi'_{\vec{a}}$ to a lower vector. In particular, suppose $a_k > a_l$, $k < l$. Then $\epsilon_l \psi'_{\vec{a}}$ will have $\psi'_{s_{k,l}\vec{a}}$ in its image. If, on the other hand, $a_k < a_l$, the terms in lines 4.13b will act by zero on those factors, and $\psi'_{s_{k,l}\vec{a}}$ will not be in the image.

Line 4.13c acts similarly—but where 4.13b acts only when indices are distinct, 4.13c acts only when indices are indistinct. The reasoning is identical, but it lowers the indices of basis vectors in two factors such that if (say) $a_k = a_l$, then $\psi_{\vec{b}}$ will appear in the image with $b_k < a_k$, $b_k = b_l$.

In summary, we have the following lemma:

Lemma 28 *The action of ϵ_l is given by*

$$\epsilon_l \psi'_{\vec{a}}(f)(1) = \lambda(\epsilon_l)\psi'_{\vec{a}}(f)(1) + \sum_{\substack{\vec{b} \leq \vec{a} \\ \vec{b} \in S_W \cdot \vec{a}}} \psi'_{\vec{b}}(f)(1) + \sum_{\vec{c} \in C} \psi'_{\vec{c}}(f)(1)$$

where

$$C = \left\{ (a'_1, \ldots, a'_n) \,\middle|\, \begin{array}{c} \exists k \text{ s.t. } a_k = a_l, a'_k = a'_l, a'_l < a_l, \\ a'_i = a_i \text{ if } i \neq k, l \end{array} \right\}$$

1. *When the indices of $\vec{a}$ are distinct, the above simplifies to:*

$$\epsilon_l \psi'_{\vec{a}}(f)(1) = \lambda(\epsilon_l)\psi'_{\vec{a}}(f)(1) + \sum_{\substack{\vec{b} \leq \vec{a} \\ \vec{b} \in S_W \cdot \vec{a}}} \psi'_{\vec{b}}(f)(1)$$

2. *When the indices of $\vec{a}$ are indistinct, the above can be rewritten:*

$$\epsilon_l \psi'_{\vec{a}}(f)(1) = (\lambda(\epsilon_l) + k(\vec{a}))\psi'_{\vec{a}}(f)(1) + \sum_{\substack{\vec{b} < \vec{a} \\ \vec{b} \in S_W \cdot \vec{a}}} \psi'_{\vec{b}}(f)(1) + \sum_{\vec{c} \in C} \psi'_{\vec{c}}(f)(1)$$

   where $k(\vec{a}) = \#\{i < l | a_i = a_l\}$.

The key fact is that the total ϵ_i action can only act to raise a vector in the ordering defined in section 4.4.1. This implies that the action is upper triangular; the weights can be read off of the diagonal.

Lemma 29 *Given a basis vector $\phi_{a_1 \ldots a_n}$, it is possible to determine the weight ξ it yields as follows.*

1. *If $a_i \neq a_j \forall j < i$, then $\xi(E_{ii}) = \lambda(E_{a_i a_i})$.*
2. *If $a_i = a_{j_1} = \cdots = a_{j_k}$ for $j_1 \ldots j_k < i$, i.e., there are k occurrences of a_i before position i, then $\xi(E_{ii}) = \lambda(E_{a_i a_i}) + k$.*

4.6 Structure of $\mathcal{F}_{\mu,\lambda}(X(\nu, \lambda))$

The key fact in this chapter is that, under appropriate conditions on λ, the dimension of $\mathcal{F}_{\mu,\lambda}(X(\nu, \lambda))$ is indeed the same as the dimension of a principal series representation of $\mathbb{H}_p \times \mathbb{H}_q$:

Lemma 30 *If $\lambda(E_{ii}) \neq \lambda(E_{jj}), \lambda(E_{jj}) \pm 1 \forall j$, then $\mathcal{F}_{\mu,\lambda}(X(\nu, \lambda))$ has dimension $p!q!$.*

Proof of lemma: Given $w \in W$, note that $w \cdot \psi_{a_1 \ldots a_n} = \psi_{w \cdot (a_1 \ldots a_n)}$. If $a_1, \ldots a_n$ are distinct, then the vectors $W \cdot \psi_{a_1 \ldots a_n}$ yield the weights $W \cdot \lambda$. Since $\lambda(E_{ii}) \neq \lambda(E_{jj}) \forall j$, all of these weights are distinct.

Since $\lambda(E_{ii}) \neq \lambda(E_{jj}) \pm 1 \forall j$, and the remaining $\psi_{a_1 \ldots a_n}$ must have indistinct $\{a_i\}$, they cannot yield any weights of the form $w \cdot \lambda$, because of the form of the weights such vectors yield.

Since the center of $\mathbb{H}_R(\mathbf{c})_{p,q} \subset \mathbb{H}_R(\mathbf{c})_n$ is $S(\mathfrak{h})^{S_p \times S_q}$, a central character for $\mathbb{H}_R(\mathbf{c})_{p,q}$ is an element of $\mathfrak{h}/S_p \times S_q$. It is thus quite easy to determine the central character of a vector in a particular weight space from its weight.

The weight spaces with $\mathbb{H}_R(\mathbf{c})_{p,q}$ central character λ will be those with weights $(S_p \times S_q) \cdot \lambda$. But by the above arguments, there is a one-dimensional weight space for each element of $(S_p \times S_q) \cdot \lambda$. In particular, then, there are $p!q!$ such one-dimensional weight spaces.
$\square$

Since the projection is on central character, the action of $\mathbb{H}_R(\mathbf{c})_{p,q}$ commutes with it; $\mathcal{F}_{\mu,\lambda}(X(\nu, \lambda))$ is thus a representation of $\mathbb{H}_R(\mathbf{c})_{p,q}$. It remains to see that it is a principal series. The following lemma is the key step.

Lemma 31 *The vectors $(S_p \times S_q) \cdot \phi_{1 \ldots n}$ form a basis of $\mathcal{F}_{\mu,\lambda}(X(\nu, \lambda))$.*

Proof of lemma: It is clear from lemma 29 that the vectors $(S_p \times S_q) \cdot \phi_{1 \ldots n}$ do yield the appropriate weights. It is also clear that the proposed basis is fixed under the action of $S_p \times S_q$. It simply remains to show that they are contained in the spans of the appropriate weight spaces.

Consider the vector $\phi_{a_1 \ldots a_n}$ in the hypothesized basis. Since $\phi_{a_1 \ldots a_n} \in (S_p \times S_q) \cdot \phi_{1 \ldots n}$, we have that $a_i \leq p \, \forall i \leq p$ and $a_i > p \, \forall i > p$. Then the result of lemma 28 can be clarified slightly. In particular, suppose

$$\epsilon_i \cdot \psi_{\vec{a}} \in \sum_{\vec{b} \in W \cdot \vec{a}} c_{\vec{b}} \psi_{\vec{b}}$$

Suppose $w \in W \setminus (S_p \times S_q)$ and $w \cdot \vec{a} = \vec{d}$. The permutation w must move elements across the break between p and $p+1$. Then there is some $i < p$ such that $d_i > p$; in that case, $\vec{d} > \vec{a}$. Then by lemma 28, $c_{\vec{d}} = 0$. Putting these facts together, we see that under the same conditions on $\psi_{\vec{a}}$ as above,

$$\epsilon_i \cdot \psi_{\vec{a}} = \sum_{\vec{b} \in (S_p \times S_q) \cdot \vec{a}} c_{i,\vec{b}} \psi_{\vec{b}}$$

It is now possible to prove the lemma by a simple induction argument. It is enough to show that the vectors $\psi_{\vec{a}}$ can all be written as sums of eigenvectors. The base case is $\psi_{1\ldots n}$; there are no lower vectors, so the equation above becomes

$$\epsilon_i \cdot \psi_{1\ldots n} = c_{i,1\ldots n}\psi_{1\ldots n}$$

In other words, $\psi_{1\ldots n}$ is itself an eigenvector.

Now suppose all vectors lower than the vector ψ can be written as sums of eigenvectors. Then constants can be chosen such that

$$\epsilon_i \cdot \psi = c_i\psi + \sum_{\xi} a_{i,\xi}\psi_{\xi}$$

where the sum is over the $S(\mathfrak{h})$-weights present in the space, and ψ_{ξ} is a specified vector in the ξ weight space. It is then easy to place a vector in the c_i eigenspace for ϵ_i. One choice is

$$\psi_c = \psi + \sum_{\xi} \frac{c_i a_{i,\xi}}{c_i - \xi(\epsilon_i)}\psi_{\xi}$$

This is the simplest way to write it down, though it clearly requires that $\xi(\epsilon_i) \neq c_i$. Fortunately this is equivalent to the existing condition that $\lambda(\epsilon_i) \neq \lambda(\epsilon_j)\,\forall i,j$. It can also be eliminated entirely simply by multiplying the entire vector by a constant. The real problem is that the above vector appears to depend on i.

In fact, by Schur's lemma we know that the representation decomposes the action of $S(\mathfrak{h})$ into simultaneous eigenspaces; the vector ψ_c is thus an eigenvector for the entire action. Then, of course, $\psi = \psi_c - \sum_{\xi} \frac{c_i a_{i,\xi}}{c_i - \xi(\epsilon_i)}\psi_{\xi}$, a sum of eigenvectors. Since there are $p!q!$ such vectors, and they are clearly linearly independent, they form a basis of $\mathcal{F}_{\mu,\lambda}(X(\nu,\lambda))$.

□

The theorem is now more or less proved.

Theorem 32 *If $\lambda(E_{ii}) \neq \lambda(E_{jj}), \lambda(E_{jj}) \pm 1 \forall j$, the space $\mathcal{F}_{\mu,\lambda}(X(\nu,\lambda))$ is a principal series representation of $\mathbb{H}_R(\mathbf{c})_{p,q}$ with infinitesimal character λ.*

Proof of theorem: From the proof of lemma 31 we know that $\mathcal{F}_{\mu,\lambda}(X(\nu,\lambda))$ contains the vector $\phi_{1\ldots n}$, which is an eigenvector for the action of $S(\mathfrak{h})$ with weight λ. By the lemma itself it is clear that this vector generates the entire space under the action of $S_p \times S_q$.

□

4.7 Further Work

The main obstruction to generalizing the results in this thesis is the case when the $K_{\mathbb{R}}$ type μ in the space

$$\mathrm{Hom}_{K_{\mathbb{R}}}\left(\mu, X \otimes (\mathbb{C}^k)^{\otimes n}\right)$$

has dimension greater than 1. Unfortunately, the metaplectic group is the only nonlinear double cover whose principal series representations have a one-dimensional minimal $K_{\mathbb{R}}$-type. This is a serious obstruction, causing the construction to fail in two separate places – showing that the action of the graded affine Hecke algebra is well-defined, and showing that the functor preserves unitarity. Fortunately the failures are similar in each case; a proof of preservation of unitarity for type A_n should help with the construction of functors for other nonlinear double covers.

Since $\mathbb{H}_p \times \mathbb{H}_q$ is a fairly manageable algebra for small p and q, it is possible to compute the unitary dual of the algebra in certain low-dimensional examples. In these cases, it seems that the existing type A_n functor does preserve unitarity; unfortunately, the general proof of preservation of unitarity is elusive.

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